The Limits of Logic: Paradoxes and The Failure of Formal Logic

Joseph Wayne Smith

Alice laughed. “There’s no use trying,” she said: “One ca’n’t believe impossible things.”

“I daresay you haven’t had much practice,” said the Queen. “When I was your age, I always did it for half-an-hour a day. Why, I sometimes believed as many as six impossible things before breakfast!” (Carroll, 1871/1988: pp. 91-92)

Indeed, if there is no formalization of logic as a whole, then there is no exact description of what logic is, for it is the very nature of an exact description that it implies a formalization. And if there is no exact description of logic, then there is no sound basis for supposing that there is such a thing as logic. (Church, 1934: p. 360)

Those who claim for themselves to judge the truth are bound to possess a criterion of truth. This criterion, then, either is without a judge's approval or has been approved. But if it is without approval, whence comes it that it is trustworthy? For no matter of dispute is to be trusted without judging. And, if it has been approved, that which approves it, in turn, either has been approved or has not been approved, and so on *ad infinitum*. (Sextus Empiricus, 1935, 179)

1. Introduction: A Skeptical Challenge to Formal Logic

In this essay, I provide an overview of the major skeptical challenges to formal or symbolic logic, both so-called classical and non-classical. The idea of formal logic, a logic of "form," will be outlined the next section, but the conclusion that I will draw is that formal logic fails to constitute some sort of formalization of correct reasoning. Indeed, we will see that the attempts by generations of formal logicians to achieve this have in fact undermined the foundations of the formal-logical endeavor. But this is not to abandon reasoning for silence or literature, for outside of Analytic philosophy, disciplines such as law, and even mathematics, get by fully adequately using informal reasoning methods, without the need for formalization, a needless burden for much of contemporary research. Rationality, and argumentation are much wider and richer fields than formal or symbolic logic per se (Hanna, 2006). And I should add in this introduction, that in striving to cover a wide field, I'm not primarily addressing this essay to the professional formal logicians who would almost certainly never abandon their position, whatever arguments are given, but rather to interested outsiders. Hence technicalities, as much as possible, will be kept to a minimum, and much use will be made of prior papers by others establishing relevant skeptical conclusions.

2. The Nature of Formal Logic

I'll now give a brief outline of formal or mathematical logic. According to Dale Jacquette:

> Logic is formal, and by itself has no content. It applies at most only indirectly to the world, as the formal theory of thoughts about and descriptions of the world. (Jacquette, 2002: p. 3)

Bertrand Russell wrote in his *Introduction to Mathematical Philosophy* that

> logic (or mathematics) is concerned with *forms*, and is concerned with them only in the way of stating that they are always or sometimes true, with all the permutations of "always" and "sometimes" that may occur. (Russell, 1919: pp. 199-200; see also Chomsky, 1975; May, 1985; Moody, 1986).

More precisely, formal deductive logic is concerned with arguments made in formal languages, which have a precise structure, a syntax, and a semantics or interpretation. The syntax defines the relationship between signs in the language and is comprised of a vocabulary, rules of formation, axioms and rules of inference. Well-formed formulas (WFFs) are specified by giving a set of symbols and rules of formation, which specifies what sequences of symbols are meaningful well-formed formulas. The semantics of a language defines the relationships between expressions in the syntax and non-linguistic objects, which collectively give an *interpretation* of the language.

Formal languages are interpreted by assigning objects (such as numbers, physical entities and so on) to the symbols and/or well-formed formulas. A formula has a *model* in a language L, if and only if there is an interpretation of the language which makes the formulae true. If X and Y are WFFs (well-formed formulas) of a language L, then Y is a *logical consequence* of X in the language L, if and only if Y is true in all models of L in which X is true. X is valid in the language if and only if X is true in all models of the language. While the concept of logical consequence is a semantic concept, the concept of proof is syntactical. A *proof* in the language is a set of well-formed formulas such that each formula is either an axiom of the language or derivable by means of the inferential rules of the system (Hunter, 1971; Kleene, 1967; Manaster, 1975). A formal logical system is *consistent* (proof-theoretically or syntactically) if and only if there is no well-formed formula X such that both X and not X (written "~X") are provable in the system. Even at this point there are major philosophical problems, such as with the concept of logical consequence and the definition of validity, but we will pass over this (Etchemendy, 1990; McGee, 1990, 1992; Priest, 1995; Gómez-Torrente).

Languages or formal logical systems are said to be *complete* if for valid arguments there is a proof in the formal system. The language or formal logical system is *sound* if no invalid arguments are provable: only proofs of valid arguments can be constructed. Some formal logical systems are incomplete, but this is not a fatal defect in the system. Unsoundness is a fatal flaw because formal deductive logic requires that if the logical form of an argument is valid, then given that its premises are true, then its conclusions must by "logical necessity" be true as well, or so the story goes (Pap, 1962: pp. 94-106). We will see that formal logic, via the paradoxes, fails here.

Most, but far from all, logicians and mathematicians accept "classical logic," which can be vaguely defined as the logic of Russell and Whitehead's *Principia Mathematica*, together with related developments. The logic is not many-valued (having three or more values to be well formed formula, such as "true", "false" and "indeterminate"), but instead has two and only two values "true" and "false" (all propositions are either true or false). Quantification—quantifiers are operators which indicate whether a statement is general (universal quantifier) or particular (existential quantifier)--occurs only over existent objects, not non-existing "objects" such as "the round square" or the "present King of the USA." Most importantly, in classical logic it is a necessary and a sufficient condition for an argument to be valid, that in every possible world (or complete interpretation), if the premises of the argument are true, then the conclusion must also be true (Read, 1988: p. 31).

The classical account of validity means that all arguments with a necessarily true conclusion, and all arguments with a necessary false conclusion or (according to classical logic) inconsistent premises, are valid. On this account, a contradiction logically implies anything. Let "&" mean "and", and "~" mean "not", then p & ~p, therefore q

is classically valid and can be proved to be so (Read, 1988). Relevant or relevance logics reject these inferences, because the logicians championing these positions believe that there should be a "logical content" connection between these premises and the conclusion of a deductively valid argument X → Y (where "→" means "implies"), where the "logical content" of the conclusion is contained in the premises (Anderson & Belnap, 1975; Routley et al., 1982; Iseminger, 1980). A movement in modern formal logic associated with relevant or relevance logics are paraconsistent logic (and mathematics) which holds that there are propositions for which X and ~X are both true, true contradictions. If there were true contradictions, and no "logical content" restrictions on logical implications, then p & ~p → q would be counter-modelled, for the premises could be true and the conclusion, an arbitrary proposition q, could be false (Priest, 2006). We will look at the significance of the so-called logico-semantical paradoxes and paraconsistency shortly in this context. It is time now to begin to examine the problems that modern formal logic and mathematics faces (Smith et al., 2023).

3. Problems with Logical Validity

According to the classical account of validity, an argument is valid if its conclusion follows from (or: is a logical consequence of, or: is logically entailed by, or: is logically implied by) its premises, and invalid if it is possible for its premises to be true and its conclusion false (or: there is some interpretation in which all its premises are true and its conclusion false). It is a necessary condition of validity that the premises of an argument cannot be true while the conclusion is false, because valid arguments are supposed to go from truth to truth, not truth to falsity (Read, 1979). The logician Stephen Read developed an argument traditionally known as the "Pseudo-Scotus," which prima facie shows the inconsistency of the concept of validity. Woodbridge and Armour-Garb have said that this paradox shows

> not just a problem with the "classical account" of validity … [but] what it shows is that our very *concept* of validity (and, thus the language we use to express it is inconsistent—at least prima facie. (Woodbridge & Armour-Garb, 2008: p. 64, 2005; Jacquette, 1996; Read, 2001)

Consider the following argument:

A: 1=1

Therefore, argument A is invalid.

To paraphrase Read's argument: suppose that argument A is valid, then A has a true premise and a false conclusion. By the classical account of validity, A is therefore invalid. Hence, if A is valid, then A is invalid. Therefore (by *reductio ad absurdum*) A is invalid. However, the premise 1=1 is a necessary truth. It is a principle of modal logic (the logic of notions such as necessity and possibility) that any proposition deduced from a necessarily true proposition, is itself necessarily true. Thus, it is necessarily true that A is invalid, and A has a necessarily true conclusion. However, on the classical account of validity (that is, the necessary truth of the conclusion of an argument is sufficient for the validity of an argument), A is valid. Therefore, A is invalid and valid: a contradiction (Read, 2001).

Another paradox can be generated with

> B: This argument is valid, therefore, this argument is invalid. (Read, 1979: p. 267)

Along similar lines it can also be shown that from these two propositions:

(I) P

and

(II) There is no sound deduction of (I) from (I) and (II)

that there is a proof that P is not true, that is, a refutation of any proposition at all! (Windt, 1973).

4. Logical Skepticism and The Problem of Deduction

Some philosophers who have attempted to solve the problem of justifying induction have argued that induction is justified because of its success and that this proposal is not question-begging because deduction itself can only be justified by deduction. Stated very roughly, deductively valid arguments are those arguments where it is logically contradictory to assert the premises and deny the conclusion, that is, it is logically impossible for the conclusion to be false and the premises true. (We have seen that there are problems even with this, the classical account of validity). Susan Haack argued in her paper "The Justification of Deduction" (Haack, 1976), that deduction faces a parallel dilemma to that which Hume raised for induction: inductive justifications of deduction will be too weak, but deductive justifications will be circular. To attempt to show the validity of the rules of inference of a formal logical system in general, would be circular in the sense of using principles of inference for which the conclusion asserts the validity of the argument (Dummett, 1973; Keene, 1975, 1983; Oakley, 1976; Strom, 1977; Bickenbach, 1979; Gallois, 1993; Fox, 1999). As Cellucci puts it: "The trouble with the standard characterization of deductive inferences is that … the proof of the validity of the rules of deductive logic is circular" (Cellucci, 2006: p. 225).

George Couvalis has also argued that we cannot know logical and mathematical truths without using experience and induction. This makes induction epistemologically prior to deduction (Couvalis, 2004). Modernizing an argument found in the work of David Hume, Couvalis says:

> To get to know a logical truth we must use an appropriately functioning entity such as a computer or a brain. Past philosophers talked about transparently infallible immediate apprehensions by the soul. But such views rely on dubious ontological assumptions and do not fit well with the fact that we sometimes make mistakes, even in simple cases. To the best of our knowledge, our minds can know logical or mathematical truths only if they at least supervene on a structured material entity, such as a brain or a computer. If it is to be reliable, this entity must function in an appropriate way. Because it is a structured material entity, it is liable to malfunction. Its malfunctions damage the power of the mental processes which it instantiates or which supervene on it. To be fairly sure it is reliable, we need ways of telling that it is functioning in an appropriate way. All such ways use inductive reasoning to reason to the conclusion that someone's brain or computer is likely to function well from knowledge that that brain or computer seems to have functioned well in the past. This implies that our knowledge that we know that reasoning is logically valid or invalid, or that axioms are true, is dependent on the cogency of inductive reasoning. That is, *if no inductive reasoning is cogent, we natural beings* [*nomologically*] *cannot know that we know any particular mathematical or logical statement to be true.* (Couvalis, 2004: p. 34)

Couvalis goes on to argue that while many logicians and philosophers believe that axioms (statements for which no proof or argument is given) and rules of inference are self-evident, there are problems with this view that were recognized by two of the founding fathers of modern mathematical logic, Bertrand Russell and Alfred North Whitehead. Russell and Whitehead said:

> [S]elf-evidence is never more than a part of a reason for accepting an axiom, and is never indispensable. The reason for accepting an axiom, as for accepting any other proposition, is always largely inductive, namely that many propositions which are nearly indubitable can be deduced from it, and that no equally plausible way is known by which these propositions could be true and the axiom were false, and nothing which is probably false can be deduced from it. … In formal logic, the element of doubt is less than in most sciences, but it is not absent, as appears from the fact that the paradoxes followed from premises which were not previously known to require limitation. (Russell & Whitehead, 1927: p. 59)

If Couvalis is right—and his arguments strike me as being correct—then there is a dilemma. Either some inductive reasoning must be accepted as valid or we should be skeptical about the justification of our knowledge of logical and mathematical knowledge. Couvalis does not deal with the resolution of this dilemma in the essay I've cited. Indications are that he is not a skeptic about logical and mathematical knowledge. But that will require a solution to the problem of justifying induction, which most philosophers grant is unsolved. Hence, deduction requires a justification as much as induction, and this problem is no closer to a solution than the solution of the problem of induction (Cellucci, 2006). However, there are other reasons for supposing that deduction can fail, producing unsoundness, indicted by the logico-semantic paradoxes.

5. The Logico-Semantical Paradoxes

Logico-semantical paradoxes are almost as old as Western philosophy (Martin, 1970). The liar paradox of Epimenides the Cretan arose from the statement "I am lying," which is true if it is false and false if it is true. A modern variant to consider is:

(L) This sentence is false.

There are "strengthened paradoxes," a sentence that says of itself that it is not true and variants of this, such as a sentence that says of itself that it is not definitely true (Mackie, 1973; Goldstein & Goddard, 1980; Parsons, 1984; McGee, 1991; Heck, 1993; Goldstein, 1994; Mills, 1995; Sorensen, 1998; Soamer, 1991; Priest, 2000a, 2000b; Greenough, 2001; Bueno & Colyvan, 2003).

The 20th century also saw the presentation of a number of other surprising paradoxes. Löb's paradox involves considering a sentence A which is true if and only if it implies B:

(L1) $A \leftrightarrow (A \rightarrow B)$.

Assume then:

(L2) A,

then

(L3) $A \rightarrow B$,

and

(L4) B. Withdraw A,

so:

(L5) A → B,

i.e.,

(L6) A,

therefore,

(L7) B. (Löb, 1955; Van Benthem, 1978: p. 50).

Closely related to this paradox is Curry's paradox which also proves an arbitrary proposition by generally accepted (that is, until the paradox was uncovered), logical principles (Curry, 1942; Irvine, 1992). This paradox does not involve negation, and can be formulated in set theoretic, property, semantic, and validity versions (Shapiro, 2013; Shapiro & Beall, 2018). An informal argument is as follows (Shapiro & Beall, 2018). Consider a sentence, "If S is true, then F." Then:

(C1) Given the assumption that S is true, then if S is true, then F.

And as well:

(C2) Given the assumption that S is true, then it is the case that S is true.

Now supposing that S is true, using *modus ponens* on the above conditional and antecedent gives:

(C3) Given the assumption that S is true, then F.

By conditional proof, the conditional can be affirmed, and assuming the antecedent yields:

(C4) If S is true, then F.

Since (C4) is S, then:

(C5) S is true.

By *modus ponens* from (C4) and (C5), then:

(C6) F, an arbitrary statement.

Better known than these paradoxes are the paradoxes of set theory such as Russell's paradox. Consider the set of all sets which are not members of themselves. Is this set a member of itself? If it is, then it is not. If it is not, then it is (Moorecroft, 1993). There are a number of other set-theoretical paradoxes such as the Burali-Forti paradox and Cantor's paradox, which need not be discussed here (Weir, 1998). Typically the set theoretical paradoxes have been dealt with by modifying our naïve conception of a set through various formal set theories. Ingenious as these theories have been it would appear from a survey of the critical literature that a final resolution of these difficulties has not been accomplished (Weir, 1998).

For example, Grim argued for some time that the set of all truths, or, all true statements, is in conflict with Cantor's power set theorem (Grim, 1984). The power set is the set of all subsets of a given set, and if a set S has *n* elements, then the Power set PS has 2^n elements (Suppes, 1960: p. 46-48). If we take the intuitive idea of a "set" to be a "collection of entities of any sort" (Suppes, 1960: p. 1; Wang, 1974: p. 181), then we should be able to deal meaningfully with both the set of all apples, and the set of all true statements. Of course, in the light of the set-theoretical paradoxes, logicians have restricted the objects of sets containing special set constituents, as in the set of all sets, and have made a distinction between sets and classes. But set theory should not yield paradoxes merely from considering elements such as sentences, which are ontologically distinct from sets. However, for the set of all truths, for each subset of this set, there will be a truth, and thus a corresponding statement, so there will be at least as many truths as there are elements of the power set, contrary to the power set theorem—or in some systems, axiom (Suppes, 1960: p. 46). Thus, a counter-example is presented to a provable theorem. Reflecting on this result and other paradoxes of totalities, Rescher and Grim state:

> Set theory was born in paradox, was shaped by paradox, and continues to carry the threat of paradox into its current adolescence. Properly understood … the threat of contradiction is not merely formal and is not to be evaded by merely formal techniques. The fact that there can be no set of all non-self-membered sets might be shrugged aside as a minor logical surprise. Beyond Russell's paradoxical set, however, there are serious philosophical difficulties of coherently conceptualising a set of all things, the realm of unrestricted quantification (or even the sense of restricted quantification), the totality of all events, all facts, all propositions, or all that is true. Sets are structurally incapable of handling any of these. (Rescher & Grim, 2011: p. 6)

Relatedly, it can be noted that the logician Wilfred Hodges once published a paper, "An Editor Recalls Some Hopeless Papers" (Hodges, 1998). This related to "crank" critiques of Cantor's diagonal argument. He wrote that almost all the "cranks" attacked the matrix representation of the sequence of decimal real numbers, but "none of the authors showed any knowledge of Cantor's theorem about the cardinalities of power sets" (Hodges, 1998: p. 2). Well, now they have the Rescher-Grim paradox.

From all this, Benson Mates concluded that

> although each possible point of contact is identified by someone as the source of the difficulty, each is also exonerated by the great majority; and consequently, no purported solution ever comes close to general acceptance. (Mates, 1981: p. 5)

Mates believed that our fundamental concepts such as set, truth etc. may be *radically defective* "in the sense, that, the clearer we get about them, the clearer it becomes that they lead to contradiction and must be repaired, if possible, or, failing that, replaced" (Mates, 1981, 57). Heck has also concluded that:

> there can be no consistent resolution of the semantic paradoxes that does not involve abandoning truth-theoretic principles that should be every bit dear to our hearts as the T-schema once was. And that leads me … to be tempted to conclude that there can be no truly satisfying, consistent resolution of the Liar paradox. (Heck, 2012, 39)

A subject dear to the hearts of popular science writers in this field is that of Gödel's incompleteness theorem (Franzen, 2005). The proof of this theorem was published by the logician Kurt Gödel in 1931. The proof showed the existence of formally undecidable propositions in certain formal systems of arithmetic. One such system of arithmetic is Peano arithmetic which has as its axioms: (1) 0 is a number; (2) the successor of any number is a number; (3) no two numbers have the same successor; (4) 0 is not the successor of any number and (5) if a predicate P is true of 0 (i.e. P(0) is true), and if it is true that $P(n) \rightarrow P(n+1)$, then P is true of all numbers. The formal theory of Peano arithmetic PA is open to Gödel's first incompleteness theorem. This states that in the formal theory PA there is a sentence G of PA such that if PA is consistent, neither G nor ~G can be proved in the formal theory. There are various ways that this theorem can be proved, with associated logical and philosophical issues (Butrick, 1965). One method involves use of a "diagonal" argument arguably similar to the liar paradox (Martin, 1977; Humphries, 1979; Johnstone, 1981). Let the Gödel sentence be the sentence:

(G) This sentence is not provable from the axioms of Peano arithmetic.

Thus, G is true, but is unprovable in PA. Suppose then, that G is not true. Then given the statement of G's contents, then G must be provable in PA. Assume that the axioms of PA are true and that the system is logically sound. Then statements provable in PA must be true. G is provable in PA. Therefore, G is true. However, the statement that G is true contradicts our initial assumption that G is not true. Therefore, G is true. If G is true, then by the Tarski principle, True (P) → P, what G says, holds, G is not provable (Barwise & Etchemendy, 2007).

There are a number of interesting papers (and chapters in books), most of them appropriately peer-reviewed, reporting some challenging ramifications of Gödel's incompleteness theorem, and its method of proof. For example, Martin (Martin, 1977) shows that a diagonal statement: (TS) "Nothing in the discourse D bears a relation R to exactly the things in the discourse D that don't bear R to themselves," underlies a number of syntactical and semantical paradoxes, as well as some important results in metalogic such as Gödel's theorem, Cantor's power-set theorem, Tarski's theorem, and every instance of the diagonal argument (Martin, 1977: p. 455). The intriguing philosophical question is how to distinguish between "good" (i.e. non-paradoxical) and "bad" (i.e. paradoxical) uses of (TS). Logical skeptics maintain that there is no method of distinguishing the "good" from the "bad" uses of (TS), so all uses are therefore regarded as problematic (Smith, 1988).

Ketland (Ketland, 2000), has proved that there is a sentence K (which "says of itself that it is not a true sentence" (Ketland, 2000: p. 1), such that K is provable in the system PA(S). PA is standard first-order Peano arithmetic in a formal language L, the first-order language of arithmetic. PA(S) is a semantical extension of PA resulting from adding a primitive satisfaction predicate $Sat_L(x, y)$. By way of explanation: an object or sequence of objects satisfies a predicate if the predicate "holds" (is "true") of the object or sequence of objects. For example, the object "electron" satisfies "does not simultaneously have definite position and momentum values," because according to mainstream quantum theory the electron does not simultaneously have definite position and momentum values. In formal semantics, the satisfaction concept is used to define a formal concept of systems-relative truth (Kaye, 1991; Ketland, 1999). Therefore, adding a primitive satisfaction predicate to PA is unobjectionable. However, Ketland shows that K, the strengthened liar formula that says of itself that it is not true, is provable in PA(S).

Graham Priest has also produced an argument demonstrating an alleged surprising consequence of Gödel's theorem (Priest, 2006a; Berto, 2009). He states Gödel's theorem as follows: let T be a theory which can represent all recursive functions and where the proof relation of T is recursive. To explain: recursive functions, are functions that can be defined from the constant, successor, and projection functions by composition of functions and recursive definitions. A recursive definition applies to the first term of a series and then for a successor term, through the predecessor of that term. To require that the proof theory of T be recursive is to require a proof be effectively recognizable, a reasonable requirement. Priest rightly observes that it is essential to the very concept of a proof, that a proof should be effectively recognizable, for the very point of a proof is to give us a way of determining whether something is true or not. Given all this, Priest states Gödel's (first) incompleteness theorem as follows: if T is consistent then there is a formula ø, Gödel's, such that (1) ø is not provable in T and (2) if the axioms and rules of inference of T are intuitively correct, then ø can be shown to be true by an intuitively correct argument. An "intuitively correct argument" refers to the type of non-formalized arguments used by mathematicians in their daily work. These methods of informal proof are generally accepted to be capable of formalization. Thus, the naïve notion of proof satisfies the conditions of Gödel's theorem.

Priest shows that the assumption of the consistency of the naïve notion of proof leads to contradiction. Let T be the formalization of the naïve theory of proof. T satisfies the conditions of Gödel's theorem. Thus, if T is consistent then there is a sentence ø which is not provable in T, but which can be shown to be true in T by a naïve proof. But the naïve notion of proof is just T, so ø is provable in T after all! (Priest, 2006 a, 39). Priest then concludes that the

> only way out of the problem, other than to accept the contradiction, and thus dialetheism [i.e., the idea that there are true contradictions] anyway, is to accept the inconsistency of naïve proof. (Priest, 2006a: p. 41)

Priest's argument was first published in a peer-reviewed journal (Priest, 1979b) and has been criticized, but defended by him (Priest, 1984). As Priest notes, the Gödel sentence is a paradoxical sentence. Informally, it is "This sentence is not provably true." Assume that the sentence is false. Then the sentence is provably true. Therefore, it is true. By *reductio ad absurdum* it is therefore true. This, however, is a proof (informally). Thus, the Gödel sentence is provably true. But if the Gödel sentence is provably true, then it is not provably true, which is contradictory! Priest speculates at this point that naïve proof procedures may therefore be essentially inconsistent because the theory is capable of giving its own semantics (semantic closure) so that the semantical paradoxes will be provable in the theory. Priest concludes that this vindicates the Kant/Hegel thesis that Reason is inherently inconsistent (Kallestrup, 2007).

Priest could be correct about Reason being inherently inconsistent. He himself does not draw a skeptical conclusion from this because he believes that paraconsistent logic can control the contradictions. The problematic contradictions are not provable falsehoods or necessarily false propositions, but *true* contradictions. So, Reason, after all is saved. But is it? Consider Priest's argument from Gödel's theorem to start with. Gödel's theorem shows that T, the formalized theory of naïve proof (intuitive mathematical proof) is inconsistent. But note that the proof of Gödel's theorem given earlier, and quoted from Priest's own presentation, presupposed that T is consistent. But by Priest's theorem, T is inconsistent, that is, it is not the case that T is consistent. Therefore, it is not the case that Gödel's theorem is correct. If Gödel's theorem is incorrect then Priest's theorem fails because it presupposes the correctness of Gödel's theorem, so that the theorem seems to be self-undermining. This does not rehabilitate classical logic, because it was classical consistency assumptions which generated this logical spiral in the first place. There is thus something intrinsically problematic with the Gödel sentence. At this point we need to look more closely at the paraconsistent approach to the logico-semantical paradoxes.

6. Paraconsistency

The paraconsistent criticism of classical logic has led to some interesting developments in metaphysics (Priest, 1999, 2000a, 2006 b; Beall, 2000; Beall & Colyvan, 2001; Kabay, 2006; Baten et al., 2000; Carnielli et al., 2002) and formally useful work in paraconsistent logic and mathematics (Priest, 1997; Mortensen, 1987, 1995), especially for automated reasoning and information processing with computer systems in which a data base contains inconsistent data (Besnard & Hunter, 1998). Needless to say, many of these useful formal developments would still be possible without accepting that there are true contradictions: all that is needed at a minimum for automated reasoning with inconsistent data is to prevent triviality occurring. So why then believe that there are true contradictions?

Priest and others generally believe that the logico-semantical paradoxes present the best case for dialetheism. The classical solutions to the paradoxes all face difficulties and something of a logician's task of Hercules:

> For every single argument they must locate a premise that is untrue, or a step that is invalid. Of course, choosing a point at which to break each argument is not difficult: we can just choose one at random. The problem is to justify the choice. It is my contention that no choice has been satisfactorily justified, and moreover, that no choice can be. (Priest, 1987: p. 11)

Presumably these remarks are made about the logico-semantical paradoxes and not *all* "logical/metaphysical" paradoxes dear to the hearts of philosophical logicians. Consider, for example, the ancient Sorites paradox, or paradox of the heap, associated with Eubulides of Miletus. This paradox can be stated as follows:

> One thousand stones, suitably arranged, might form a heap. If we remove a single stone from a heap of stones we still have a heap; at no point will the removal of just one stone make sufficient difference to transform a heap into something which is not a heap. But, if this is so, we still have a heap, even when we have removed the last stone composing our original structure. (Burgess, 1990: p. 417)

The argument need not use the concept of a "heap" but can still be restated with any number of vague predicates. Thus, 0 is a small number. If n is a small number, then

$n+1$ is a small number. Therefore, by the principle of mathematical induction, all numbers are small (Priest, 1979a: pp. 74-75). As Priest has said: "Mathematical induction is shown to be an invalid form of argument when fuzzy properties are involved" (Priest, 1979a: p. 75). The Sorites paradox can be generated by finitely many applications of *modus ponens* (if p then q, p, therefore q) or by use of the substitutivity of identicals. So, according to Priest's claim about mathematical induction, these logical principles too are invalid when fuzzy/vague properties are considered (Dummett, 1975; Lakoff, 1973; Hanfling, 2001; Keefe & Smith, 1999). If not, why not?

The mathematician Florentin Smarandache has produced a number of quantum mechanics sorites paradoxes (Smarandache, 2005). For example, there is not a clear dichotomy between matter which on the large-scale behaves deterministically, and matter which is subject to Heisenberg's indeterminacy principle (variables specifying the position and momentum of subatomic particles cannot simultaneously both take definitive values). In general, philosophers have paid insufficient attention to the Smarandache paradoxes. No matter: vagueness has already "become a philosopher's nightmare" (Napoli, 1985: p. 115; Russell, 1923; Schwartz & Throop, 1991). In a survey of solutions to the sorites paradox Richard De Witt says that "all the proposals offered to date as ways of blocking the paradox are seriously deficient" (DeWitt, 1992: pp. 93-94). Priest is also of the opinion that "no extant solution to the Sorites paradox works" (Priest, 1991: p. 296)—and that presumably includes a paraconsistent solution. If one is to postulate that situations of vagueness involve true contradictions, then much of the observable world would be contradictory, a position which Priest does not embrace (Priest, 2006a).

The thesis that taking the paradoxes as being sound arguments delivering a true conclusion (a true contradiction) constitutes a unified and non-ad hoc solution to the logico-semantical paradoxes, is also contestable (Everett, 1993, 1994; Mares, 2000; Beall, 2001; Bromand, 2002). Curry's paradox and Löb's paradox, for example, do not have a "true contradiction" as a conclusion, but rather an arbitrary proposition. Correspondingly, some have argued that paraconsistent logics still face the Curry/Löb paradoxes (Everett, 1996). Many paraconsistent logics are reduced to triviality from Curry-style paradoxes (Slaney, 1989; Restall, 2007). One response to this has been to reject the principle of absorption:

$$\text{(AB)} \quad (A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B),$$

read as "If A implies A implies B, then A implies B." This has involved the alleged construction of a countermodel to (AB). Even so, Geach has shown how a sentence A such that $A \rightarrow (A \rightarrow B)$, where B is an arbitrary statement can be constructed (Geach, 1955).

Priest's response to Curry's paradox, in his system LP, is to reject the general validity of *modus ponens* (Carrara & Martino, 2011: p. 200). *Modus ponens* fails, Priest proposes, when dialetheic sentences that are both true and false, occur. It has been argued in reply that this denial is ad hoc, and as well, Curry's paradox can be derived without *modus ponens*, using only the naïve notion of deducibility, which Priest accepts (Carrara & Martino, 2011: p. 203-205). Hence Priest's dialetheism does not avoid trivialism (Carrara et al., 2010).

Armour-Garb and Woodbridge have constructed pathological sentences that defy classical and paraconsistent responses, they alleged (Armour-Garb & Woodbridge, 2006). An example is:

(C1) (C2) is true→ 'Everything is true'

(C2) (C1) is true→ 'Everything is true'

The "open pair " (in the above case "Curried open pair") has a simpler form:

(1) (2) is false

(2) (1) is false,

which generates a pathological oscillation. Amour-Garb and Woodbridge argue, convincingly in my opinion, that both consistent solutions and paraconsistent solutions to the "open pair" paradox fail. Debate continues on this issue.

It is to be expected that the ultimate result of all this logical research would be ruin. To begin, the dialetheist position, that some contradictions are true, represented as "D is true" where D is a dialetheia (a true contradiction) turns out on Priest's account of paraconsistency, LP, to be a dialetheia itself, that is, true *and* false. Thus, the very statement of the position of *strong* paraconsistency (dialetheism) is contradictory. It has been shown that the principle of non-contradiction in Priest's system, LP, is both valid and invalid (Heald, 2016). Priest accepts this result (Priest, 1979b). One could argue, as Manuel Bremer does in his excellent *Lectures on Paraconsistent Logic*, that it is a minimum condition for the assertability of a thesis that it should be *true only* (Bremer, 2004: p. 205); no doubt dialetheists would counter this by arguing that it begs the question against them because after all they have asserted their thesis, it is open to criticism (e.g. the production of "hypercontradictions" or triviality) and so on. However, arguably, if dialetheism is true *and* false, then this position is to be rejected in favor of a position which offers a non-ad hoc unified solution to the logico-semantical paradoxes and is arguably true only. As we have seen, paraconsistency fails to provide a simple unified solution to the paradoxes in any case. Classical logic is not such position because of the arguments outlined by its paraconsistent critics. Their critique of classical logic holds, even if, which we believe is the case, paraconsistent logic has its own internally destructive problems.

7. The Refutation of Formal Logic

Modern developments in formal logic have resulted in an almost unbounded power to construct exotic counter-examples and counter-models to refute once cherished logical principles. Paraconsistent logician Chris Mortensen has said on this point:

> One of the directions of recent logical research has been into semantical conditions under which various propositions hold and fail. One of the upshots has been a growing body of information about how to construct models to refute more and more propositions. It is, for example, no news that countermodels can be constructed to large numbers of theorems of the very natural modal logic S5, on which David Lewis' modal realism is based. It is also a straightforward matter to construct countermodels to the laws of excluded middle and noncontradiction. Recent work by Errol Martin has even shown how to construct countermodels to every instance of $A \rightarrow A$. In light of these kinds of results, it seems to me that it would be a bold claim that there is *any* proposition that cannot be made to come out *false* in some structure (Mortensen, 1981: p. 57, 1989).

Countermodels to every instance of $A \rightarrow A$? Mortensen goes on to argue that given Martin's counter-modelling of $A \rightarrow A$ (in a weak propositional calculus) one can in principle doubt the seeming logical necessity of statements such as "If Smith is a bachelor then Smith is an unmarried man." That statement presupposes that "If Smith is a bachelor then Smith is a bachelor" is also necessary, which is of course a substitution instance of $A \rightarrow A$ (Mortensen, 1989: p. 329). On this basis, Mortensen says, we can conceive how our mathematics could be false:

> [It] seems to me that the intuitive solidity of mathematics rests on the same foundation. Short, quite obvious inferences in mathematics often derive, like the previous bachelor case, from some definitional decision to use terms interchangeably applied to $A \rightarrow A$, (or to $(A\&B) \rightarrow A$ or $A \rightarrow (A \vee B)$). Mathematical connections established by longer chains of reasonings appealing to more complex deductive principles are to that extent less evidently necessary. I am not suggesting here that it is *easy* to understand how standard mathematics might have been false. But then we should beware of projecting the limitations of our imaginations onto the world. (Mortensen,1989: p. 329)

Routley and Meyer have constructed relevant logic semantics where any formula of the form $x \rightarrow y$ may fail (Martin, 1978; Martin & Meyer, 1982; Routley et al., 1982). Priest agrees that the countermodels can be constructed to any arbitrary formula (Priest, 1992). According to Priest:

> [The] prime notion of logic is inference; and valid (deductive) inferences are expressed by statements of entailment, $\alpha \rightarrow \beta$, (that α entails that β). Hence in a logically impossible world we should expect statements of this form to take values other than the correct ones. Is there a limit to the value that such a conditional might take? I do not see why. Just as we can imagine a world where the laws of physics are arbitrarily different, indeed, an anomalous world where there are no such laws; so we can imagine worlds where the laws of logic are arbitrarily different, indeed an anomalous world where there are no such laws. (Priest, 1992: pp. 292-293)

Relatedly, the late Richard Sylvan (formerly Richard Routley), developed a theory of items based upon the ideas of logician Alexius Meinong (Routley, 1980). Items are everything that can be the object of thought, and things which cannot, such as: if I is defined as I = that object which is not an item, then it is an item. One can thus speculate about a prime number p between 11 and 13 or even an infinite number of prime numbers between 11 and 13, even though standard Peano arithmetic has no such p (Sylvan, nd). This position as stated has some logical difficulties including a problem with absolute inconsistency (i.e., it allows the derivation that 1=0), which Priest has addressed and which we need not discuss here (Priest, 2005). Given all this logical freedom it is seemingly inevitable that counter-examples to one's most prized principles and counter-arguments to beloved arguments would multiply "in a way that makes the breeding habits of rabbits look like family planning" (Priest, 1987, 145-146).

8. Conclusion: Bankruptcy, Non-Formalism, Limits, and Humility

The conclusion reached here is the same one I reached 40 years ago: formal logic is bankrupt: there are no "laws of form" (Smith, 1984). The same conclusion was reached by the late Australian philosopher David Stove. Stove said in his book *The Rationality of Induction:*

> There are no logical forms, above a low level of generality … There are few or no logical forms, above a low level of generality, of which every instance is valid: nearly every such supposed form has invalid cases or paradoxical cases. The natural conclusion to draw is that formal logic is a myth and that over validity, as well over invalidity, forms do *not* rule: cases do. (Stove, 1986: p. 127)

More recently, Hofweber has considered that there are counter-examples to all the inference rules, so the rules are not strictly valid, but are only valid over some range (Hofweber, 2007). The idea that formal logic has its limitations has been expressed before, of course (Rohatyn, 1974; Kekes, 1982; Devlin, 1997), but the full skeptical ramifications have seldom been embraced. Clearly, if the most precise area of human knowledge has numerous "black holes" of reason, we can expect paradoxes a-plenty in every other field, and that is exactly what we find. The existence of these unsolved logical and semantic paradoxes challenges the rationality of science, since science depends upon mathematics, and mathematics, being a so-called deductive science, crucially depends upon logic. But, if the foundations of logic are insecure, so too will be the foundations of mathematics. At this point mathematicians, who were probably Platonists before confronting the paradoxes, are likely to become pragmatists, saying that their concern is with merely making deductions from axioms, without concern about the ultimate truth and justification of them. The skeptic would be pleased to accept this, replying that if this is so, mathematics is not epistemologically different from the rest of human knowledge, where at the end of the day, pragmatism rules.

In general, we have seen that the logico-semantical paradoxes remain unsolved, even by the paraconsistency school which has taken the paradoxical sentences to be "true contradictions." And, even if the paraconsistent school is right about the limits of classical logic, their own position faces crippling objections, namely that they do not escape *all* the paradoxes, so that they therefore fail to produce a satisfactory general response to the logical challenge of the paradoxes, which the once radical move of positing "true contradictions" was supposed to solve. For the most precise of all sciences, this is indeed a major epistemological king hit. It is ironic, that increased technical sophistication in formal logic has led to a type of process of self-undermining, where all former "logical truths" and once taken-for-granted principles, such as even *modus ponens*, face counter-examples (McGee, 1985). An epistemological skeptic would see this as a major objection to the rationality of the discipline itself, and a major epistemological crisis that seemingly is intractable, at least from the perspective of Analytic philosophy.

However, as an alternative, the arguments given here can be taken to show the limits of the Analytical philosophical framework, and the need to move to a non-formalist approach to logic, as has been previously explored by Robert Hanna (Hanna, 2006), and today, by many others in the informal logic schools. Moreover and finally, this situation makes a strong case for philosophical limitationism and epistemic humility, according to which as Rae Langton puts it, "[t]here are inevitable constraints on what we can know, inevitable limits on what we can become acquainted with" (Langton, 1998: p. 2).

REFERENCES

(Anderson & Belnap, 1975). Anderson, A.R. & Belnap, N.D. *Entailment*. Princeton NJ: Princeton Univ. Press. Vol. 1.

(Armour-Garb & Woodbridge, 2006). Armour-Garb, B. & Woodbridge, J.A. "Dialetheism, Semantic Pathology, and the Open Pair." *Australasian Journal of Philosophy* 84: 395-416.

(Barwise, & Etchemendy, 2007). Barwise, J. and Etchemendy, J. *Language, Proof and Logic*. Stanford CA: CSLI Publications.

(Batens, 2000). Batens, D. et al. (eds.) *Frontiers of Paraconsistent Logic*. Baldock, Hertfordshire UK: Research Studies Press.

(Beall & Colyvan, 2001). Beall, J.C. & Colyvan, M. "Looking for Contradictions." *Australasian Journal of Philosophy* 79: 564-569.

(Beall, 2000). Beall, J.C. "Is the Observable World Consistent?" *Australasian Journal of Philosophy* 78: 113-118.

(Beall, 2001). Beall, J.C. "Dialetheism and the Probability of Contradiction." *Australasian Journal of Philosophy* 79: 114-118.

(Beall, 2007). Beall, J.C. (ed.) *Revenge of the Liar: New Essays on the Paradox*. Oxford: Oxford Univ. Press.

(Berto, 2009). Berto, F. "The Gödel Paradox and Wittgenstein's Reasons." *Philosophia Mathematica* 17: 208-209.

(Besnard & Hunter, 1998). Besnard, P & Hunter, A. (eds.) *Handbook of Defeasible Reasoning and Uncertainty Management Systems. Vol.2: Reasoning with Actual and Potential Contradictions.* Dordrecht: Kluwer.

(Bickenbach, 1979). Bickenbach, J.E. "Justifying Deduction." *Dialogue* 18: 500-516.

(Bremer, 2005). Bremer, M. *Lectures on Paraconsistent Logic*. Berlin: Peter Lang.

(Bromand, 2002). Bromand, J. "Why Paraconsistent Logic Can Only Tell Half the Truth." *Mind* 111: 741-749.

(Bueno & Colyvan, 2003). Bueno, O. & Colyvan, M. “Yablo’s Paradox and Referring to Infinite Objects.” *Australasian Journal of Philosophy* 81: 402-412.

(Burgess, 1990). Burgess, J.A. “The Sorites Paradox and Higher-Order-Vagueness.” *Synthese* 85: 417-474.

(Butrick, 1965). Butrick, R. “The Gödel Formula: Some Reservations.” *Mind* 74: 411-414.

(Carnielli, 2002). Carnielli, W.A. et al. (eds.) *Paraconsistency: The Logical Way to the Inconsistent*. New York: Marcel Dekker.

(Carrara et al. 2010). Carrara, M. et al. “Can Priest’s Dialetheism Avoid Trivialism?” In M. Pelis & V. Puncochar (eds.), *The Logica Year Book*. Pp. 1-12.

(Carrara & Martino, 2011). Carrara, M. & Martino, E. “*Curry’s Paradox*: A New Argument for Trivialism.” *Logic & Philosophy of Science* 9, 1: 199-206.

(Carroll, 1871/1988). Carroll, L. *Through the Looking-Glass*. New York: Dial.

(Cellucci, 2006). Cellucci, C. “The Question Hume Didn’t Ask: Why Should We Accept Deductive Inferences?” Demonstrative and Non-Demonstrative Reasoning in Mathematics and Natural Science Workshop, University of Rome La Sapienza. Pp. 207-235.

(Chomsky, 1975). Chomsky, N. “Questions of Form and Interpretation.” *Linguistic Analysis* 1: 75-109.

(Church, 1934). Church, A. “The Richard Paradox.” *American Mathematical Monthly* 41: 356-361.

(Couvalis, 2004). Couvalis, G. “Is Induction Epistemologically Prior to Deduction?” *Ratio* 17: 28-44.

(Curry, 1942). Curry, H.B. “The Inconsistency of Certain Formal Logics.” *Journal of Symbolic Logic* 7: 115-117.

(Devlin, 1997). Devlin, K. *Goodbye Descartes: The End of Logic and the Search for a New Cosmology of the Mind*. New York: John Wiley and Sons.

(DeWitt, 1992). DeWitt, R. “Remarks on the Current Status of the Sorites Paradox.” *Journal of Philosophical Research* 17: 93-118.

(Dummett, 1973). Dummett, M. “The Justification of Deduction.” *Proceedings of the British Academy* 59: 201-232.

(Dummett, 1975). Dummett, M. “Wang’s Paradox.” *Synthese* 30: 301-324.

(Etchemendy, 1990). Etchemendy, J. *The Concept of Logical Consequence*. Cambridge MA: Harvard Univ. Press.

(Everett, 1993). Everett, A. “A Note on Priest’s “Hypercontradictions.” *Logique et Analyse* 141-142: 39-43.

(Everett, 1994). Everett, A. “Absorbing Dialetheia.” *Mind* 103: 413-419.

(Everett, 1996). Everett, A. “A Dilemma for Priest’s Dialetheism?” *Australasian Journal of Philosophy* 74: 657-668.

(Fox, 1999). Fox, J. “Deductivism Surpassed.” *Australasian Journal of Philosophy* 77: 447-464.

(Franzen, 2005). Franzen, T. *Gödel’s Theorem: An Incomplete Guide to its Use and Abuse.* Wellesley MA: A.K. Peters.

(Gallois, 1993). Gallois, A. “Is Global Scepticism Self-Refuting?” *Australasian Journal of Philosophy* 71: 36-46.

(Geach, 1955). Geach, P.J. “On *Insolubilia*.” *Analysis* 15: 71-72.

(Goldstein & Goddard, 1980). Goldstein, L. & Goddard, L. “Strengthened Paradoxes.” *Australasian Journal of Philosophy* 58: 211-221.

(Goldstein, 1994). Goldstein, L. “A Yabloesque Paradox in Set Theory.” *Analysis* 54: 223-227.

(Gómez-Torrente, 1999). Gómez-Torrente, M. “Logical Truth and Tarskian Logical Truth.” *Synthese* 117: 375-408.

(Greenough, 2001). Greenough, P. “Free Assumptions and the Liar Paradox.” *American Philosophical Quarterly* 38: 115-135.

(Grim, 1984). Grim, P. “There is No Set of All Truths.” *Analysis* 44: 206-208.

(Haack, 1976). Haack, S. “The Justification of Deduction.” *Mind* 85: 112-119.

(Hanfling, 2001). Hanfling, O. "What is Wrong with Sorites Arguments?" *Analysis* 61: 29-35.

(Hanna, 2006). Hanna, R. *Rationality and Logic*. Cambridge MA: MIT Press. Also available online in preview at URL = <https://www.academia.edu/21202624/Rationality_and_Logic_MIT_Press_2006_>.

(Heald, 2016). Heald, G. "Why LP Paraconsistent Logic is Paradoxical." Available online at URL = <https://www.researchgate.net/publication/301341710>.

(Heck, 1993). Heck, R. "A Note on the Logic of Higher Order Vagueness." *Analysis* 53: 201-208.

(Heck, 2012). Heck, R. "A Liar Paradox." *Thought* 1: 36-40.

(Hodges, 1998). Hodges, W. "An Editor Recalls Some Hopeless Papers." *Bulletin of Symbolic Logic* 4: 1-16.

(Hofweber, 2007). Hofweber, T. "Validity, Paradox, and the Ideal of Deductive Logic." In (Beall, 2007: pp. 145-158).

(Humphries, 1979). Humphries, J. "Gödel's Proof and the Liar Paradox." *Notre Dame Journal of Formal Logic* 20: 535-544.

(Hunter, 1971). Hunter, G. *Metalogic: An Introduction to the Metatheory of Standard First Order Logic*. London: Macmillan.

(Irvine, 1992). Irvine, A.D. "Gaps, Gluts, and Paradox." *Canadian Journal of Philosophy*. Supplementary Vol. 18: 273-299.

(Iseminger, 1980). Iseminger, G.I. "Is Relevance Necessary for Validity?" *Mind* 89: 196-213.

(Jacquette, 1996). Jacquette, D. "The Validity Paradox in Model S5" *Synthese* 109: 47-62.

(Jacquette, 2002). D. Jacquette, D. "Introduction: Logic, Philosophy, and Philosophical Logic." In D. Jacquette (ed.), *A Companion to Philosophical Logic*, Oxford: Blackwell. Pp. 1-8.

(Johnstone, 1981). Johnstone, A.A. "Self-Reference, the Double Life and Gödel." *Logique et Analyse* 24: 35-47.

(Kabay, 2006). Kabay, P. "When Seeing is Not Believing: A Critique of Priest's Argument from Perception." *Australasian Journal of Philosophy* 84: 443-460.

(Kallestrup, 2007). Kallestrup, J. "If Omniscient Beings are Dialetheists, then So are Anti-Realists." *Analysis* 67: 252-254.

(Kaye, 1991). Kaye, R. *Models of Peano Arithmetic*. Oxford: Clarendon/Oxford Univ. Press.

(Keefe & Smith, 1999). Keefe, R. & Smith, P. (eds.) *Vagueness: A Reader*. Cambridge MA: MIT Press.

(Keene, 1975). Keene, G.B. "On the Logic of the Circularity of Logic." *Mind* 84: 100-101.

(Keene, 1983). Keene, G.B. "Self-Referent Inference and the Liar Paradox." *Mind* 92: 430-433.

(Kekes, 1982). Kekes, J. "Logicism." *Idealistic Studies* 12: 1-13.

(Ketland, 1999). Ketland, J. "Deflationism and Tarski's Paradise." *Mind* 108: 69-94.

(Ketland, 2000). Ketland, J. "A Proof of the (Strengthened) Liar Formula in a Semantical Extension of Peano Arithmetic." *Analysis* 60: 1-4.

(Kleene, 1967). Kleene, S. C. *Mathematical Logic*. New York: John Wiley & Sons.

(Lakoff, 1973). Lakoff, G. "Hedges: A Study in Meaning Criteria and the Logic of Fuzzy Concepts." *Journal of Philosophic Logic* 2: 458-508.

(Langton, 1998). Langton, R. *Kantian Humility: Our Ignorance of Things in Themselves*. Oxford: Oxford University Press.

(Löb, 1955). Löb, M. H. "Solution of a Problem of Leon Henkin." *Journal of Symbolic Logic* 20: 115-119.

(Mackie, 1973). Mackie, J.L. *Truth, Probability and Paradox: Studies in Philosophical Logic*. Oxford: Clarendon/Oxford Univ. Press.

(Manaster, 1975). Manaster, A. B. *Completeness, Compactness, and Undecidability: An Introduction to Mathematical Logic*. Englewood Cliffs NJ: Prentice-Hall.

(Mares, 2000). Mares, E. D. “Even Dialetheists Should Hate Contradictions.” *Australasian Journal of Philosophy* 78: 503-516.

(Martin & Meyer, 1982). Martin, E. P. & Meyer, R. K. “Solution to the P-W Problem.” *Journal of Symbolic Logic* 47: 869-887.

(Martin, 1970). Martin, R.L. (ed.) *The Paradox of the Liar*. New Haven CT: Yale Univ. Press.

(Martin, 1977). Martin, R. L. “On a Puzzling Classical Validity.” *Philosophical Review* 86: 454-473.

(Martin, 1978). Martin, E.P. *The P-W Problem*. PhD dissertation. Australian National University.

(Mates, 1981). Mates, B. *Skeptical Essays*, Chicago & London: Univ. of Chicago Press.

(May, 1985). May, R. *Logical Form: Its Structure and Derivation*. Cambridge MA: MIT Press.

(McGee, 1985). McGee, V. “A Counterexample to Modus Ponens.” *Journal of Philosophy* 82: 462-471.

(McGee, 1990). McGee, V. “Review of Etchemendy (1990).” *Journal of Symbolic Logic* 57: 254-255.

(McGee, 1991). McGee, V. *Truth, Vagueness and Paradox*. Indianapolis IN: Hackett.

(McGee, 1992). McGee, V. “Two Problems with Tarski’s Theory of Consequence.” *Proceedings of the Aristotelian Society* 92: 273-292.

(Mills, 1995). Mills, A. “Unsettled Problems with Vague Truth.” *Canadian Journal of Philosophy* 25: 103-117.

(Moody, 1986). Moody, T. “The Indeterminacy of Logical Forms.”*Australasian Journal of Philosophy* 64: 190-205.

(Moorecroft, 1993). Moorecroft, F. “Why Russell’s Paradox Won’t Go Away.” *Philosophy* 68: 99-103.

(Mortensen, 1981). Mortensen, C. “A Plea for Model Theory.” *Philosophical Quarterly* 31: 152-157.

(Mortensen, 1987). Mortensen, C. “Inconsistent Nonstandard Arithmetic.” *Journal of Symbolic Logic* 52: 512-518.

(Mortensen, 1989). Mortensen, C. “Anything is Possible.” *Erkenntnis* 30: 319-337.

(Mortensen, 1995). Mortensen, C. *Inconsistent Mathematics*. Dordrecht: Kluwer.

(Napoli, 1985). Napoli, E. “Is Vagueness a Logical Enigma?” *Erkenntnis* 23: 115-121.

(Oakley, 1976). Oakley, T. “An Argument for Skepticism about Justified Beliefs.” *American Philosophical Quarterly* 13: 221-228.

(Pap, 1962). Pap, A. “The Laws of Logic.” In A. Pap, *An Introduction to the Philosophy of Science*. New York: Free Press. Pp. 94-106.

(Parsons, 1984). Parsons, T. “Assertion, Denial and the Liar Paradox.” *Journal of Philosophic Logic* 13: 137-152.

(Priest, 1979a). Priest, G. “A Note on the Sorites Paradox.” *Australasian Journal of Philosophy* 57: 74-75.

(Priest, 1979 b). Priest, G. “The Logic of Paradox.” *Journal of Philosophic Logic* 8: 219-241.

(Priest, 1984). Priest, G. “Logic of Paradox Revisited.” *Journal of Philosophic Logic* 13: 153-179.

(Priest, 1987). Priest, G. “Unstable Solutions to the Liar Paradox.” In S.J. Bartlett, and P. Suber (eds.), *Self-Reference: Reflections on Reflexivity* Dordrecht: Martinus Nijhoff. Pp. 145-175.

(Priest, 1987). Priest, G. *In Contradiction: A Study of the Transconsistent*. Dordrecht: Martinus Nijhoff.

(Priest, 1991). Priest, G. “Sorites and Identity.” *Logique et Analyse* 34 : 293-296.

(Priest, 1992). Priest, G. “What is a Non-Normal World?” *Logique et Analyse* 35 : 291-302.

(Priest, 1995). Priest, G. “Etchemendy and Logical Consequence.” *Canadian Journal of Philosophy* 25: 283-292.

(Priest, 1997). Priest, G. “Inconsistent Models of Arithmetic: Part I: Finite Models.” *Journal of Philosophical Logic*. 26: 223-235.

(Priest, 1999). Priest, G. “Perceiving Contradictions.” *Australasian Journal of Philosophy* 77: 439-446.

(Priest, 2000a). Priest, G. “Could Everything be True?” *Australasian Journal of Philosophy* 78: 189-195.

(Priest, 2000b). Priest, G. “Truth and Contradiction.” *Philosophical Quarterly* 50: 305-319.

(Priest, 2005). Priest, G. *Towards Non-Being: The Logic and Metaphysics of Intentionality*. New York: Oxford University Press.

(Priest, 2006a). Priest, G. *In Contradiction: A Study of the Transconsistent*. 2nd edn., Oxford: Clarendon/Oxford Univ. Press.

(Priest, 2006b). Priest, G. *Doubt Truth to be a Liar*. Oxford: Clarendon Press.

(Read, 1979). Read, S. “Self-Reference and Validity.” *Synthese* 42: 265-274.

(Read, 1988). Read, S. *Relevant Logic: A Philosophical Examination of Inference* Oxford: Basil Blackwell.

(Read, 2001). Read, S. “Self-Reference and Validity Revisited.” In M. Yrjönsuuri (ed.), *Medieval Formal Logic: Obligations, Insolubles, and Consequences*. Dordrecht: Kluwer. Pp. 183-196.

(Rescher & Grim, 2011). Rescher, N. & Grim, P. *Beyond Sets: A Venture in Collection-Theoretic Revisionism*. Germany: Ontos Verlag.

(Restall, 2007). Restall, G. “Curry’s Revenge: The Costs of Non-Classical Solutions to the Paradoxes of Self-Reference.” In (Beall, 2007: pp. 261-271).

(Rohatyn, 1974). Rohatyn, D.A. “Against the Logicians: Some Informed Polemics.” *Dialectica* 28: 87-102.

(Routley, 1980). Routley, R. *Exploring Meinong’s Jungle and Beyond*. Canberra AU: Department of Philosophy, Research School of Social Sciences, Australian National University.

(Routley, 1982). Routley, R. et al. *Relevant Logics and their Rivals I*. Atascadero CA: Ridgeview.

(Russell, 1919). Russell, B. *Introduction to Mathematical Philosophy*. London: George Allen and Unwin.

(Russell, 1923). Russell, B. "Vagueness." *Australasian Journal of Philosophy*. 1: 84-92.

(Russell & Whitehead, 1927). Russell, B. & Whitehead, A. N. *Principia Mathematica I.* 2nd edn., Cambridge: Cambridge Univ. Press.

(Schwartz & Throop, 1991). Schwartz, S.P. and W. Throop, W. "Intuitionism and Vagueness." *Erkenntnis* 34: 347-356.

(Sextus Empiricus, 1935). Sextus Empiricus. *Against the Logicians.* Trans. R.G. Bury. London: W. Heinemann.

(Shapiro, 2013). Shapiro, L. "Validity Curry Strengthened." *Thought* 2: 100-107.

(Shapiro & Beall, 2018). Shapiro, L & Beall. J.C. "Curry's Paradox." In E.N. Zalta (ed.), *Stanford Encyclopedia of Philosophy*. Available online at URL = < <https://plato.stanford.edu/entries/curry-paradox/>.

(Slaney, 1989). Slaney, J. "RWX is Not Curry Paraconsistent." In G. Priest et al., *Paraconsistent Logic: Essays on the Inconsistent*. Munich: Philosophia. Pp. 472-480.

(Smarandache, 2005). Smarandache, F. "Quantum Quasi-Paradoxes and Quantum Sorites Paradoxes." *Progress in Physics* 1 : 7-8.

(Smith, 1984). Smith, J.W. "Formal Logic: A Degenerating Research Programme in Crisis." *Cogito* 2, 3: 1-18.

(Smith, 1986). Smith, J.W. *Reason, Science and Paradox*: *Against Received Opinion in Science and Philosophy*. London: Croom Helm.

(Smith, 1988). Smith, J.W. "The Illogic of Logic: The Paradoxes and the Crisis of Modern Logic." In J.W. Smith, *Essays on Ultimate Questions: Critical Discussions of the Limits of Contemporary Philosophical Inquiry*. Aldershot UK: Avebury. Pp. 124-176.

(Smith, 1999). Smith, J.W. "Fingernails on the Mind's Blackboard: Universal Reason, Postmodernity and the Limits of Science." In J.W. Smith et al., *The Bankruptcy of Economics*. London: Macmillan. Pp. 55-58.

(Smith et al., 2023). Smith, J.W., Smith, S., & Stocks, N. "Gödel's Theorems, the (In)Consistency of Arithmetic, and the Fundamental Mistake of Analytic Philosophers of Mathematical Logic. *Against Professional Philosophy*. 10 September. Available online at URL = <https://againstprofphil.org/2023/09/10/godels-theorems-the-inconsistency-of-arithmetic-and-the-fundamental-mistake-of-analytic-philosophers-of-mathematical-logic/>.

(Soames, 1999). Soames, S. *Understanding Truth*. Oxford: Oxford Univ. Press.

(Sorensen, 1998). Sorensen, R.A. "Yablo's Paradox and Kindred Infinite Liars." *Mind* 107: 137-155.

(Stove, 1986). Stove, D.C. *The Rationality of Induction*. Oxford: Clarendon/Oxford Univ. Press.

(Strom, 1977). Strom, J.J. "On Squaring Some Circles of Logic." *Analysis* 37: 127-129.

(Suppes, 1960). Suppes, P. *Axiomatic Set Theory*. Princeton NJ: D. Van Nostrand.

(Sylvan, n.d.). Sylvan, R. "Item Theory Further Liberalized." Unpublished MS.

(van Benthem, 1978). van Benthem, J.F.A.K. "Four Paradoxes." *Journal of Philosophic Logic* 7: 49-72.

(Wang, 1974). Wang, H. *From Mathematics to Philosophy*. London: Routledge & Kegan Paul.

(Weir, 1998). Weir, A. "Naïve Set Theory is Innocent!" *Mind* 107: 763-798.

(Windt, 1973). Windt, P.Y. "The Liar in the Prediction Paradox." *American Philosophical Quarterly* 10: 65-68.

(Woodbridge & Armour-Garb, 2005). Woodbridge, J.A. and Armour-Garb, B. "Semantic Pathology and the Open Pair." *Philosophy and Phenomenological Research* 71: 695-703.

(Woodbridge & Armour-Garb, 2008). Woodbridge, J.A. & Armour-Garb, B. "The Pathology of Validity." *Synthese* 160: 63-74.