The Limits of Limits: A Skeptical Inquiry into the Foundations of the Calculus

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ε-δ Definition

$$\lim_{x \to a} f(x) = L \text{ if } \forall \epsilon > 0$$

$$\exists \delta > 0 \text{ such that}$$

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$$

In geometric terms, this definition stipulates that if the function values **f(x)** must be within an interval centered around **L**, then it is always possible to find an interval centered around **x = a** that ensures this condition. (BL, 2023)

1. Introduction

In this essay, we critically examine the standard ε–δ definition of limits as formulated by Karl Weierstrass (1815-1897), arguing that its claim to foundational rigor masks a series of unresolved conceptual problems. While the formal apparatus successfully avoided vague notions of infinitesimals (for the time, but now formalized in treatments such as nonstandard analysis (Robinson, 1966; Bell, 1998; Stroyan and Luxemburg, 1976; Davis, 1977) and Smooth Infinitesimal Analysis (SIA) (Moerdijk and Reyes,1991)) and dynamic approximation, it does so at the cost of introducing new forms of abstraction that are no less philosophically contentious. We explore two central objections: (i) the "infinite tasks" problem, which questions the operational meaning of universal quantification over all ε > 0, and (ii) the "circularity" objection, which highlights the definitional dependence on a pre-assumed limit value L. Drawing on critiques by N. J. Wildberger and J. Gabriel (Gabriel, n.d.), we investigate whether these issues reflect deep structural flaws in the epistemic status of limits, continuity, and differentiation. An alternative approach, such as that based on convergent sequences, is assessed and found to either replicate or reframe the same foundational concerns. Our critical analysis suggests that modern limit theory, while indispensable in practice, may rest on conceptual foundations that are provisionally coherent rather than intrinsically rigorous. We conclude by considering whether the pursuit of formal precision in analysis has, paradoxically, obscured the intuitive and ontological ambiguities it sought to resolve.

2. The Formal Definition of Limits: Weierstrass and the Alleged Achievement of Mathematical Rigor

The development of calculus in the 17^{th} and 18^{th} centuries produced powerful computational tools, but it rested on philosophically troubling foundations (Cauchy, 1821; Boyer, 1949; Kline, 1972; Grabiner, 1981; Lakoff and Núñez, 2000; Ehrlich, 2006). The widespread use of infinitesimals—quantities treated as both zero and non-zero depending on mathematical convenience—drew sharp criticism from observers like Bishop Berkeley, who derided these "ghosts of departed quantities" as logically incoherent. Even when mathematicians abandoned infinitesimals in favor of more intuitive language about limits "tending toward" values, fundamental questions remained unanswered. What did it mean, precisely, for a mathematical quantity to "approach" something? How could mathematics claim logical rigor while relying on metaphorical descriptions borrowed from motion and time (Błaszczyk et al., 2013)?

Karl Weierstrass revolutionized mathematical analysis by providing an answer that eliminated all appeals to intuition and established calculus on a foundation of pure logical precision, allegedly (Weierstrass, 1894). His epsilon-delta definition of limits transformed the field not merely through technical innovation, but also by demonstrating how mathematical concepts could achieve complete rigor without sacrificing their essential meaning, supposedly.

The Weierstrass definition, as per the diagram at the top of this essay, states that the limit of $f(x)$ as x approaches a equals L if and only if: for every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$. This deceptively simple statement contains profound philosophical implications embedded within its logical structure.

The definition operates by means of what might be called a "challenge-response framework." The epsilon represents a challenge: no matter how small a positive number someone chooses as a measure of acceptable accuracy, the limit claim must be defensible. And the delta provides the response: given any such challenge, there exists a corresponding restriction on the input variable that guarantees the desired level of accuracy in the output. Crucially, this relationship must hold universally—for every possible epsilon, no matter how small.

Let's consider the geometric interpretation of this logical structure. The epsilon creates a horizontal band of width 2ε centered around the proposed limit value L, extending from (L - ε) to (L + ε). This represents the acceptable "tolerance" for how close function values must be to the limit. Correspondingly, δ creates a vertical band of width 2δ centered around the approach point a, extending from (a - δ) to (a + δ), while explicitly excluding the point x = a itself through the condition $0 < |x - a|$. The definition requires that whenever the input x falls within this δ-neighborhood of a, the corresponding function value f(x) must fall within the ε-neighborhood of L.

What makes this definition philosophically revolutionary is its complete elimination of dynamic or temporal language. There is no reference to variables "approaching" values, no description of "tending toward" limits, no invocation of processes unfolding over time. Instead, Weierstrass replaced all such metaphorical language with precise logical relationships between numerical quantities. The limit is not something that happens; it's a property that either holds or fails to hold based on the existence of appropriate δ values for all possible ε challenges.

This transformation addressed the fundamental philosophical problem that had plagued earlier approaches to limits. When mathematicians spoke of variables "approaching" limits, they imported intuitions from physical motion into pure mathematics. But mathematical objects have no location in space and undergo no processes in time. By recasting limits as static logical relationships, Weierstrass demonstrated that mathematical analysis could achieve complete precision without relying on potentially misleading physical analogies.

The logical structure also clarifies exactly what must be proven to establish the existence of a limit. To demonstrate that a limit exists, one must show that for any proposed ε—no matter how small—a suitable δ can be found that satisfies the required relationship. Contrapositively, to prove that a limit does not exist, one need only identify a single epsilon value for which no corresponding delta can guarantee the necessary relationship. This transforms limit proofs from vague arguments about "getting arbitrarily close" into precise logical demonstrations.

The Weierstrass definition represents more than just a technical improvement in mathematical analysis; it exemplifies a fundamental shift in the philosophical understanding of mathematical rigor. Prior approaches to limits, whether through infinitesimals or intuitive "tending" language, retained connections to non-mathematical intuitions about physical processes or spatial relationships. Weierstrass demonstrated that mathematics could achieve autonomy from such external supports.

This achievement established a template that influenced the development of modern mathematical standards throughout analysis and beyond. The definition showed how careful logical formulation could eliminate conceptual confusion while preserving the essential mathematical content that made calculus so powerful. Rather than weakening mathematical intuition, the rigorous approach actually strengthened it by providing a solid foundation upon which geometric and physical insights could be confidently built.

The epsilon-delta definition also answered Berkeley's critique by showing that calculus required no "ghosts of departed quantities" or other logically problematic entities. Every element of the definition refers to ordinary real numbers and their relationships. The apparent mystery of limits—how could something approach a value without reaching it?—dissolved once limits were understood as logical properties rather than dynamic processes.

3. The Triumph of Rigor?

Weierstrass's contribution demonstrates that rigor need not come at the expense of mathematical power or intuition. By providing calculus with a logically unassailable foundation, the epsilon-delta definition actually enhanced the field's capacity for both theoretical development and practical application. The definition established that mathematical truth could be determined through purely logical criteria, independent of physical analogies or temporal metaphors.

A Worked-Out Example: Not Trivial

Consider the limit of the following expression as x approaches 2:

$$\lim (x \to 2) [\sqrt{(x^2 + 5)} - 3] / [x^2 - 4]$$

Initial Analysis: Why This Limit is Non-Obvious

At first glance, this limit presents several challenges that make its behavior far from intuitive:

Direct Substitution Fails: If we attempt to substitute $x = 2$ directly, we get:

Numerator: $\sqrt{(2^2 + 5)} - 3 = \sqrt{9} - 3 = 3 - 3 = 0$

Denominator: $2^2 - 4 = 4 - 4 = 0$

This gives us the indeterminate form 0/0, which provides no information about the limit's value.

Geometric Intuition is Unclear

Unlike simple polynomial ratios, the combination of a square root function in the numerator with a quadratic in the denominator creates a relationship whose behavior near $x = 2$ is not immediately apparent from graphical reasoning.

Standard Techniques Don't Directly Apply

This is neither a simple rational function nor a standard trigonometric limit, so common limit theorems don't directly apply.

The Solution: Rationalization Strategy

The key insight is to eliminate the square root from the numerator through rationalization—multiplying both numerator and denominator by the conjugate expression.

Step 1: Identify the Conjugate: The conjugate of $(\sqrt{(x^2 + 5)} - 3)$ is $(\sqrt{(x^2 + 5)} + 3)$.

Step 2: Apply Rationalization: $\lim (x \to 2) [\sqrt{(x^2 + 5)} - 3] / [x^2 - 4] \times [\sqrt{(x^2 + 5)} + 3] / [\sqrt{(x^2 + 5)} + 3]$

Step 3: Simplify the Numerator: Using the difference of squares formula $(a - b)(a + b) = a^2 - b^2$:

Numerator = $[\sqrt{(x^2 + 5)} - 3][\sqrt{(x^2 + 5)} + 3] = (\sqrt{(x^2 + 5)})^2 - 3^2 = (x^2 + 5) - 9 = x^2 - 4$

Step 4: Rewrite the Expression: $\lim (x \to 2) (x^2 - 4) / [(x^2 - 4)(\sqrt{(x^2 + 5)} + 3)]$

Step 5: Cancel Common Factors: Since $x \neq 2$ in the limit process, we can cancel $(x^2 - 4)$ from numerator and denominator:

$\lim (x \to 2) 1 / [\sqrt{(x^2 + 5)} + 3]$

Step 6: Evaluate the Simplified Limit: Now direct substitution works: $= 1 / [\sqrt{(2^2 + 5)} + 3] = 1 / [\sqrt{9} + 3] = 1 / [3 + 3] = 1/6$

Verification and Interpretation

The Answer: $\lim (x \to 2) [\sqrt{(x^2 + 5)} - 3] / [x^2 - 4] = 1/6$

Why This Result is Significant: Its Hidden Simplicity

What appeared to be a complex rational expression with a square root actually simplifies to a basic fraction evaluation.

Cancellation Reveals Structure

The indeterminate form 0/0 masked the fact that both numerator and denominator contained the factor $(x^2 - 4)$, which created the apparent singularity at $x = 2$.

Technique Dependence

Without the rationalization technique, this limit would be extremely difficult to evaluate; L'Hôpital's Rule could have been used, but this presupposes calculus. The method transforms an opaque expression into a transparent one.

Algebraic vs. Geometric Insight

While graphical analysis might eventually reveal the limit value, the algebraic manipulation provides both the exact answer and insight into why this particular value emerges.

Broader Mathematical Lessons

This example illustrates several important principles about limits and mathematical analysis. **First**, indeterminate forms require techniques: The 0/0 form signals that deeper analysis is needed, not that the limit fails to exist. **Second**, algebraic manipulation reveals hidden structure: The rationalization technique exposed the underlying mathematical relationship that direct substitution could not reveal. **Third**, non-obvious limits often have simple answers: complex-looking expressions frequently simplify to elegant results when approached with appropriate techniques. **Fourth**, the power of conjugates: rationalization using conjugate expressions is a powerful tool for handling limits involving square roots, particularly when they produce indeterminate forms.

This worked-out example demonstrates how mathematical rigor, embodied in systematic algebraic techniques, can reveal precise answers to questions that remain opaque to direct intuition or simple computational approaches.

4. Objections to the Standard Theory of Limits

4.1 The Skeptical Challenge: Wildberger's Critique of Epsilon-Delta Rigor

Despite the apparent triumph of Weierstrass's epsilon-delta definition in establishing mathematical rigor, the approach has not been without its critics. Among the most prominent contemporary skeptics is Norman Wildberger, whose systematic critique challenges both the pedagogical effectiveness and philosophical foundations of the classical limit definition. Wildberger's objections represent a significant challenge to the standard narrative of mathematical progress, questioning whether the epsilon-delta approach truly represents an unqualified improvement over earlier methods. His critique is spread out over the past decade on his YouTube channel, rather than in academic papers, his aim being to reach a wider audience. One can listen to the lectures at YouTube, best found by using the YouTube search engine. "Logical Difficulties with the Modern Theory of Limits (I) and (II)," is a good summary.

The Intuition Problem

Wildberger's primary criticism targets what he sees as the counterintuitive nature of the epsilon-delta definition. While classical mathematicians celebrate the elimination of vague language about "approaching" or "tending toward," Wildberger argues this formal precision comes at the cost of conceptual clarity. Students, he contends, find themselves memorizing technical procedures without developing genuine understanding of what limits actually represent. The definition requires manipulating arbitrary small positive numbers epsilon and delta according to logical rules that, while formally correct, provide little insight into the underlying mathematical phenomenon (Tall, 1980; White, 2005).

This pedagogical concern extends beyond mere teaching difficulties. Wildberger suggests that the epsilon-delta approach fundamentally misconceives how mathematical understanding develops. By prioritizing formal manipulation over geometric intuition, the definition may actually impede rather than enhance mathematical comprehension. Students learn to verify that specific delta values work for given epsilon challenges, but this technical skill does not necessarily translate into deeper insight about functional behavior or the nature of continuity.

The "Infinite Tasks" Problem

Perhaps Wildberger's most philosophically challenging argument concerns what he characterizes as the "infinite tasks" problem inherent in the epsilon-delta definition. The requirement that "for every epsilon greater than zero, there exists a delta" seems to demand verification of infinitely many conditions simultaneously.

How can mathematicians claim to have established a limit when the definition appears to require completing an impossible infinite checklist?

This objection, if correct, strikes at the heart of mathematical practice. When mathematicians prove that a limit equals a particular value, they typically demonstrate a general method for finding appropriate delta values given any epsilon, rather than literally checking every possible epsilon. But Wildberger questions whether this general demonstration truly establishes what the definition claims to establish. The gap between the logical structure of the definition—which quantifies over all positive real numbers—and the finite proofs mathematicians actually construct suggests a fundamental tension in the classical approach.

Geometric Intuition versus Formal Abstraction

Wildberger advocates approaches that prioritize geometric visualization and spatial reasoning over abstract logical manipulation. The epsilon-delta definition, he argues, is essentially analytic rather than geometric, focusing on numerical relationships rather than the visual and spatial intuitions that often guide mathematical discovery. This emphasis on formalism over geometry may obscure rather than illuminate the mathematical relationships that limits are meant to capture.

The historical irony here is significant. While the epsilon-delta definition was developed to eliminate the perceived inadequacies of intuitive approaches, Wildberger suggests that mathematical intuition—properly channelled through geometric reasoning—might provide more reliable and accessible foundations than abstract logical formalism. He proposes that alternative approaches, such as those based on sequences and convergence, could offer more intuitive pathways to understanding limits while maintaining mathematical precision.

4.2 The Classical Response

Classical mathematicians, however, offer substantial counterarguments to Wildberger's critique. They maintain that formal precision, far from being a pedagogical hindrance, provides the necessary foundation for reliable mathematical reasoning. The epsilon-delta definition's apparent abstraction actually represents a crucial intellectual achievement: the successful translation of intuitive ideas into logically rigorous forms that can withstand critical scrutiny.

As regards the infinite tasks problem, classical mathematicians argue that Wildberger misunderstands the logical structure of mathematical quantification. The definition does not require performing infinitely many separate verifications; rather, it establishes a logical relationship that must hold universally. When mathematicians prove a limit exists, they demonstrate that this relationship can be satisfied, not that they have checked every possible case individually. The universality of the quantification is a feature of the logical structure, not a practical impossibility.

Furthermore, classical mathematicians contend that the epsilon-delta approach has demonstrated its value through more than a century of successful application in mathematical analysis. The definition has enabled the development of sophisticated theories in real analysis, complex analysis, and functional analysis that would be impossible without rigorous foundational concepts. While alternative approaches may offer pedagogical advantages in certain contexts, they typically rely on the same underlying logical principles that the epsilon-delta definition makes explicit.

4.3 The Deeper Philosophical Tension

Wildberger’s critique ultimately raises fundamental questions about the nature of mathematical knowledge and the relationship between formal rigor and mathematical understanding. His challenge suggests that the mathematical community’s emphasis on formal precision may have led to approaches that are technically correct but conceptually impoverished. The epsilon-delta definition, while logically unassailable, may fail to capture essential aspects of how mathematicians actually think about limits and continuity.

This tension reflects broader philosophical questions about whether mathematical truth is primarily formal or intuitive, whether rigor should be valued above insight, and whether the evolution toward greater abstraction represents genuine progress or a movement away from mathematics’ essential character (Lakatos, 1976). Wildberger’s skeptical challenge thus serves not merely as a critique of a particular definition, but as a fundamental questioning of the values and priorities that guide contemporary mathematical practice.

5. The Circularity Objection: A Fundamental Challenge to Epsilon-Delta Logic

While Wildberger's critique focused primarily on pedagogical and intuitive concerns, a more fundamental logical challenge to the epsilon-delta definition emerges from critics like John Gabriel (Gabriel, n.d), who argue that the definition suffers from a basic circularity that undermines its claim to rigorous foundations (Boyer, 1949: p. 281). This objection strikes at the heart of the epsilon-delta approach by questioning whether the definition can coherently establish what it purports to define.

The Logical Problem with Circular Definitions

At the heart of the skeptical challenge to the epsilon-delta definition of limits lies a question of logical legitimacy: what is wrong with a definition that refers to itself? Why is circularity, in a definitional context, considered such a serious flaw?

The answer lies in the fundamental role that definitions play in logical and mathematical systems. A proper definition is meant to provide a *reduction*: it explains a concept in terms of previously understood or more basic concepts. Definitions establish the foundation from which proofs, theorems, and further reasoning can proceed. If a definition includes the very term it is attempting to define—explicitly or implicitly—it fails to reduce; instead, it merely restates or smuggles in the undefined notion through the back door.

This is not a matter of stylistic purity—it's a breach of epistemic hierarchy. In formal systems, we rely on well-founded chains of meaning, where terms are built from simpler, already defined components. A circular definition breaks that chain and collapses the hierarchy, rendering the system vulnerable to ambiguity, arbitrariness, or contradiction.

To use a classic analogy: defining "truth" as "a true statement" tells us nothing about *what* truth is. Similarly, defining the limit L of a function as the value that the function gets arbitrarily close to—*by comparing it to L*—amounts to saying "the limit is the limit."

Some forms of circularity may be benign or even necessary in informal language—for example, mutual recursion in computer science or interdependent concepts in natural language. But in foundational mathematics, where the entire edifice of formal reasoning is built on clarity and unambiguous definitions, circularity undermines the integrity of deduction itself.

Moreover, circular definitions are unprovable by design. If a definition presupposes the thing to be defined, any attempted proof becomes a tautology dressed up in technical language. This results in what philosophers of mathematics sometimes call a "pseudo-proof"—a statement that feels rigorous only because it hides its assumptions in the very terms it purports to establish.

If the goal of foundational mathematics is to build upwards from clear and independent primitives, then a circular definition is not just a minor technical lapse. It is a foundational failure—a violation of the very principles mathematical rigor claims to uphold.

6. The Structure of the Circularity

As we've seen, the standard epsilon-delta definition of limits states that the limit of f(x) as x approaches a equals L if and only if: for every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$. The circularity objection centers on a seemingly obvious but troubling feature of this definition: the limit value L appears within the definition itself. The definition presupposes the existence of the very quantity it claims to define.

This circularity operates at multiple levels. **First**, the definition requires us to know L in order to verify whether the epsilon-delta conditions are satisfied. We cannot determine whether $|f(x) - L| < \varepsilon$ without already having determined what L is. And **second**, the logical structure suggests that we are defining "the limit L" by reference to L itself—a form of circular reasoning that would be rejected in other contexts as fundamentally flawed.

Consider the parallel with attempting to define "the tallest person in the room" by saying "the tallest person in the room is the person P such that everyone else in the room is shorter than P." This definition presupposes that we already know who P is, making it useless for actually identifying the tallest person. Similarly, the epsilon-delta definition seems to presuppose knowledge of L while claiming to establish what L is.

The Foundational Implications

If the circularity objection is valid, it suggests profound problems with the mathematical foundations that have been built upon epsilon-delta definitions. The entire edifice of real analysis, with its sophisticated theorems about continuity, differentiability, and convergence, would rest upon a logically flawed foundation. This is not merely a technical quibble but a fundamental challenge to the coherence of modern classical mathematical analysis, although, of course, not alternatives such as nonstandard analysis.

The circularity objection also raises questions about what mathematicians are actually doing when they "prove" that a limit exists. If the definition is circular, then limit proofs cannot establish the existence of limits in the way they claim to. Instead, these proofs might be better understood as demonstrations that certain algebraic manipulations are possible, or that certain patterns hold, without actually establishing the existence of the mathematical objects they purport to define.

Furthermore, the objection suggests that the historical triumph over infinitesimals and intuitive approaches may have been pyrrhic. While the epsilon-delta definition eliminated the alleged logical problems of earlier approaches, it may have introduced an even more fundamental logical flaw—circularity—that renders the entire enterprise questionable.

Cascading Effects: How Circular Definitions of Limits Undermine Classical Differential Calculus

If the ε-δ definition of limits contains a fundamental logical circularity, the implications extend far beyond abstract mathematical theory. The entire edifice of differential calculus, built upon the foundation of limits, would inherit these logical flaws, creating a cascade of foundational problems throughout mathematics and its applications.

Differential calculus stands on the concept of limits as its primary foundation. The derivative—calculus's central operation—is defined as the limit of a difference quotient:

$$f'(x) = \lim[h \to 0] \ [f(x+h) - f(x)]/h$$

This definition makes every derivative calculation dependent on the logical soundness of the limit concept. If limits suffer from circular reasoning, then derivatives, as constructs built entirely upon limits, inherit this circularity wholesale.

The circularity pervades calculus's fundamental concepts. Continuity relies entirely on limit definitions. A function f is continuous at point c if $\lim[x \to c] \ f(x) = f(c)$. Any logical problems with limits immediately compromise our understanding of continuity, affecting everything from basic function analysis to advanced topology.
Differentiability compounds the problem by requiring both limits and continuity. Since differentiable functions must be continuous, and derivatives are defined as limits, differentiability inherits circularity from both sources.

Compromise of Major Theorems

The foundational theorems of calculus become logically suspect. The Mean Value Theorem states that for a continuous function on [a,b] that is differentiable on (a,b), there exists some c where f'(c) equals the average rate of change. Since this theorem depends on both continuity and differentiability—both limit-dependent concepts—circular limit definitions would undermine its logical foundation.

Rolle's Theorem, the Fundamental Theorem of Calculus, and L'Hôpital's Rule all similarly depend on limit-based definitions. Each would require re-examination if limits prove circular.

7. Applications Under Threat

The practical consequences extend beyond pure mathematics. Physics relies heavily on derivatives for concepts like velocity, acceleration, and electromagnetic field calculations. If derivative definitions are circular, it raises questions about the mathematical foundations underlying physical laws. Engineering applications from structural analysis to control systems depend on calculus-based models. Circular foundations could compromise the theoretical basis for these critical applications. Economics uses calculus for optimization problems, marginal analysis, and dynamic modelling. Logical circularity in limits would affect the mathematical rigor of these economic theories.

The Circularity Inheritance Problem

The core issue is that mathematical definitions should be built from more basic, well-defined concepts. If limits are defined circularly, then every concept built upon them—derivatives, integrals, continuity, differentiability—inherits this circularity. This creates a house of cards whereby the entire structure depends on a conceptually flawed foundation.

The mathematical community has generally dismissed circularity concerns about limits, but if such concerns prove valid, they would necessitate a complete reconstruction of analysis. This would require either (i) developing alternative, non-circular definitions of limits, as constructionists have done, or (ii) finding entirely different approaches to derivatives and continuity, which for example, nonstandard analysis and Smooth Infinitesimal Analysis (SIA) allegedly do.

Accepting that current classical calculus operates on logically imperfect foundations is the worst of these options. The implications of circular limit definitions extend far beyond abstract mathematical philosophy. They strike at the heart of one of mathematics' most successful and widely-applied theories. While calculus undeniably works in practice, the question of whether its theoretical foundations are logically sound remains critical for mathematical rigor and the long-term development of analysis.

Rejoinders: The Cultural Practice Defense and Its Limitations

Classical mathematicians typically respond to the circularity objection by arguing that the definition's widespread acceptance and practical utility demonstrate its validity. They contend that mathematical definitions need not conform to strict logical purity as long as they enable productive mathematical work. The epsilon-delta definition has facilitated centuries of successful mathematical development, and this pragmatic success allegedly justifies its use despite potential logical concerns.

However, this defense faces serious philosophical challenges. The appeal to cultural practice and historical success commits what might be called the "pragmatic fallacy"—the assumption that usefulness implies truth or logical validity. Many historically successful but ultimately false theories in science and mathematics demonstrate that practical utility does not guarantee logical soundness. The geocentric model of planetary motion, for instance, was enormously useful for astronomical calculations while being fundamentally incorrect about the structure of the solar system.

Moreover, the cultural practice defense fails to address the specific logical problem that has been identified. If the definition is genuinely circular, then its acceptance by the mathematical community might reflect collective oversight rather than collective wisdom. Mathematical communities have, historically, accepted definitions and approaches that were later recognized as problematic or inadequate.

The Deeper Skeptical Challenge

The circularity objection connects to broader philosophical skepticism about mathematical knowledge and foundations. Drawing on David Hume's problem of induction, mathematical skeptics can argue that the acceptance of epsilon-delta definitions represents a form of inductive reasoning that cannot be logically justified. The mathematical community has inductively concluded that these definitions are sound based on their past utility, but this inductive inference cannot provide the kind of logical certainty that mathematics claims to achieve.

This skeptical perspective suggests that mathematical definitions and concepts are ultimately human constructions subject to the same uncertainties and fallibilities that characterize other human endeavors. The search for absolute logical rigor in mathematics may be fundamentally misguided, and different definitional approaches may coexist without any being inherently superior to others.

The existence of alternative approaches to limits—such as sequence-based definitions or Cauchy's formulation—supports this skeptical view. If multiple, potentially incompatible approaches can serve as foundations for calculus, this suggests that the choice among them is more a matter of convention and convenience than logical necessity. The epsilon-delta approach's dominance might reflect only historical contingency rather than logical superiority.

Implications for Mathematical Education and Practice

If the circularity objection is taken seriously, it has significant implications for how mathematics should be taught and understood. Rather than presenting the epsilon-delta definition as the uniquely rigorous foundation for calculus, educators might need to acknowledge its limitations and present it as one approach among several possible alternatives. This would require a more modest and less dogmatic approach to mathematical foundations.

The objection also suggests that mathematical skepticism serves a valuable critical function by exposing potential weaknesses in accepted approaches. Even if the circularity objection does not ultimately undermine the epsilon-delta definition, it forces mathematicians to examine and defend their foundational assumptions more carefully.

The circularity objection thus presents a fundamental challenge that cannot be easily dismissed through appeals to practical utility or historical precedent. If the objection is valid, it entails that the epsilon-delta definition fails to provide the logical rigor it claims to establish. If the objection is invalid, classical mathematicians need to provide more convincing logical arguments for why the apparent circularity does not constitute a genuine problem.

This unresolved tension points to deeper questions about the nature of mathematical definition, the relationship between logical rigor and practical utility, and the extent to which mathematics can achieve the kind of certainty and foundation that mathematicians have traditionally claimed. Our critique, pushed to its full skeptical conclusion, suggests that mathematical foundations may be far more provisional and problematic than the mathematical community has generally acknowledged.

8. An Alternative Definition: The Sequence-Based Approach and Its Limitations

In response to concerns about the ε-δ definition of limits, mathematicians have developed several alternative approaches (Rudin, 1976; Abbott, 2001; Bartle and Sherbert, 2011). While nonstandard analysis using infinitesimals represents one significant alternative that merits separate detailed critical examination, the sequence-based definition has also gained particular attention and traction as a potentially more intuitive foundation for limit theory.

The sequence-based approach defines limits through convergent sequences rather than the ε-δ framework. Under this definition, a function f has a limit L at point c if, for every sequence $\{x_n\}$ converging to c (where $x_n \neq c$ for all n), the sequence $\{f(x_n)\}$ converges to L.

This definition appears to offer several advantages: it can be more intuitive for students familiar with sequence convergence; it avoids the quantifier complexity of the ε-δ definition; and it connects function limits directly to the more elementary concept of sequence limits.

But despite its apparent advantages, the sequence-based definition faces significant objections that suggest it may not resolve the foundational issues plaguing the ε-δ approach.

The Circularity Problem Persists

The most serious objection concerns circularity. The sequence-based definition relies fundamentally on the concept of sequence convergence, which itself requires a rigorous definition. When we examine how sequence convergence is typically defined, we find it depends on the same ε-δ framework we sought to avoid: a sequence $\{x_n\}$ converges to L if for every $\varepsilon > 0$, there exists N such that for all $n > N$, $|x_n - L| < \varepsilon$.

This creates a circular dependency: we define function limits using sequence limits, but sequence limits are defined using the same ε-δ logic. Rather than eliminating the foundational problems, the sequence-based approach merely shifts them to a different level of abstraction.

Loss of Geometric Intuition

The ε-δ definition, whatever its logical flaws, provides a clear geometric interpretation by means of an appeal to neighborhoods and distances. The sequence-based definition lacks this intuitive spatial understanding. While sequences can be visualized, the connection between sequence convergence and function behavior at a point becomes more abstract, potentially making the concept less accessible rather than more so.

Complications with Directional Limits

The sequence-based definition becomes cumbersome when handling one-sided limits or directional approaches. While the ε-δ framework naturally accommodates left-hand and right-hand limits through appropriate modifications, the sequence-based approach requires additional constructs and qualifications that complicate rather than simplify the definition.

Multivariable Complexity

In higher dimensions, the sequence-based definition becomes significantly more complex and less intuitive. The ε-δ approach generalizes naturally to functions of multiple variables, but the sequence-based approach struggles with the variety of ways sequences can approach a point in multidimensional spaces. This limitation makes it less suitable as a foundational framework for advanced analysis.

Philosophical and Foundational Concerns

From a foundational perspective, the sequence-based definition faces similar philosophical objections to those raised against the ε-δ approach. The concept of sequence convergence still relies on the notion of “arbitrarily close” or “approaching” a limit, which contains the same conceptual circularities that plague other definitions.

Moreover, constructivist and finitist mathematicians (Bridger and Richman, 1987; Bishop, 1967) raise additional concerns: the sequence-based definition relies on infinite sequences and their completion, concepts that may not be constructively valid. If one objects to the ε-δ definition on grounds of infinite processes or non-constructive elements, the sequence-based approach faces similar objections.

Pedagogical Challenges

Contrary to claims that the sequence-based approach is more intuitive, many students find the abstract nature of sequences and their convergence difficult to grasp. The connection between sequence behavior and function limits can be less obvious than the direct geometric relationship expressed in the ε-δ definition. This suggests that the purported pedagogical advantages of the sequence –based approach might be overstated.

The Inadequacy of Alternative Approaches

The examination of the sequence-based definition reveals a troubling pattern: alternative approaches to limits often relocate rather than resolve the fundamental logical issues. Whether we use sequences, nets, filters, or other topological constructs, we consistently encounter similar problems of circularity, infinite processes, and foundational assumptions.

This in turn suggests that the problems with limit definitions may be more fundamental than they initially seemed. Rather than representing flaws in a particular definitional approach, they may reflect deeper issues with how we conceptualize the notion of mathematical “approach” or being “arbitrarily close.”

Implications for Foundational Reform

The failure of alternative definitions to resolve circularity concerns has significant implications. If multiple independent approaches to limits suffer from similar logical problems, it suggests that the issue lies not with specific definitions but with our fundamental conceptual framework for understanding limiting behavior.

This points toward the need for more radical foundational reforms rather than definitional modifications. Such reforms might require either abandoning traditional approaches to limits entirely, or developing completely new mathematical frameworks that avoid the problematic concepts of "approaching" or "arbitrarily close," which has been done with nonstandard analysis, for example.

The sequence-based definition, while offering some practical advantages in specific contexts, ultimately fails to provide the foundational clarity needed to resolve the logical concerns surrounding limit theory. Its examination reveals that the problems with limits may be more deeply rooted in our mathematical conceptual framework than has been previously recognized.

9. Conclusion: The Illusion of Rigor?

The epsilon-delta definition of limits has long been enshrined as the pinnacle of mathematical precision—a triumph of logic over intuition, and rigor over ambiguity. But as we've shown in this this essay, that triumph may be less secure than it appears. What is celebrated as a foundation may instead be a carefully concealed circle: a definition that assumes what it claims to define, a standard that requires the impossible execution of infinite tasks, and a framework that obscures mathematical meaning behind layers of abstraction.

Critics like Wildberger have exposed the philosophical cracks in this classical mathematical structure, but the mathematical community has largely responded with silence, dismissal, or appeals to tradition and utility. Yet pragmatic success is not the same as conceptual coherence. If we accept definitions on the basis of habit or effectiveness rather than clarity and justification, then mathematics risks becoming a kind of ritualized formalism—fluent in symbols but detached from meaning.

Attempts to reform limit theory, via sequences, or constructivist logic, have typically replicated the same foundational problems they aimed to solve. This suggests that the difficulty lies not in the choice of formalism, but in the very concept of "approach" itself—a concept suspended uneasily between static logic and dynamic intuition. Perhaps it is not calculus that needs rescuing, but our philosophy of mathematics that needs rethinking, as an important minority have done?

The path forward is unclear, but necessary. We may need to rethink what we mean by rigor, to re-engage with geometric intuition, or to develop finitist or constructive alternatives that avoid the metaphysical sleights of hand buried in classical analysis. What cannot continue is the uncritical veneration of a definition that refuses to define without borrowing what it seeks to prove.

If the price of rigor is circularity, then what exactly are we proving?

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