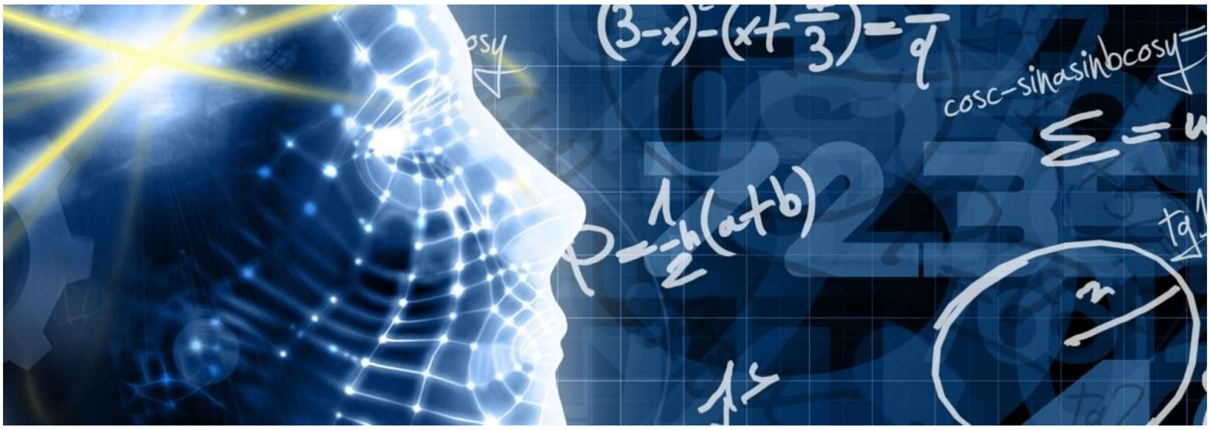
Mathematics, Metaphysics, and Mystery

Joseph Wayne Smith and N. Stocks



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> In perpetuating these notions, modern mathematics takes on many of the aspects of a *religion*. It has its essential creed – namely Set Theory, and its unquestioned assumptions, namely that mathematics is based on "Axioms," in particular the Zermelo-Frankel "Axioms of Set Theory." It has its anointed priesthood, the *logicians*, who specialize in studying the *foundations of mathematics*, a supposedly deep and difficult subject that requires years of devotion to master. Other mathematicians learn to invoke the official mantras when questioned by outsiders, but have only a hazy view about how the elementary aspects of the subject hang together logically. (Wildberger, 2015: p. 2)

1. Introduction: From Mind to Mathematics

Anyone who has an interest in the philosophy of mind and consciousness, would be acutely aware that mind and consciousness pose problems to the received "scientific" worldview, one where phenomena need to fit into the world as described by physics, chemistry and biology, the natural sciences which define the scope and limits of reality. Specifically, consciousness is a problem for a mechanistic worldview, one which holds that:

> [*E*]*verything in the world is fundamentally either a formal automaton or a natural automaton*, operating strictly according to Turing-computable algorithms and/or time-reversible or time-symmetric deterministic or indeterministic laws of nature, especially the Conservation Laws (including the First Law of Thermodynamics) and the Second Law of Thermodynamics, which also imposes always-increasing entropy—i.e., the always-increasing unavailability of any system's thermal energy for conversion into causal (aka "mechanical") action or work—on all natural mechanisms, until a total equilibrium state of the natural universe is finally reached. (Hanna, 2024: p. 23)

It is a problem, the so-called "hard problem of consciousness," to explain the subjective and qualitative aspects of human (and animal) consciousness within this physicalist/mechanistic materialist framework, that is, why there is "something it is like" to have a conscious experience (Nagel, 1974; Chalmers, 1996). We could conceive of philosophical "zombies," being exactly like us, physically, but lacking conscious experience. Hence the physical account of conscious seems to be incomplete, but for those who embrace a mechanistic world view, this is a scandal, that consciousness itself becomes something of a metaphysical mystery (McGinn, 1991, 2000; Levine, 2001).

In this essay, we will argue that the concern with consciousness being a "mystery" within a mechanistic/physicalist framework is *not* a problem, since there is an even more troublesome problem of metaphysical mystery right in the heart of this framework. While there is a case that the very idea of the physical has its own metaphysical puzzlement, the issue discussed here concerns abstract essence, particularly the ontology of mathematical entities. It will be argued that there is a more fundamental problem of accounting for these within the mechanistic/physicalist framework, and indeed, within any framework. Hence, we are neck deep in "mysteries," and up against the limits of the scientific world view, so we should not be unduly concerned that consciousness remains unexplained. There are more unexplained things in "heaven" and earth, Horatio, than are dreamt of in the scientific mechanistic/materialist worldview, and since this framework has mathematics as its basis, metaphysical mystery here undercuts the quest to reduce everything in reality, to the physically explicable.

2. The Nature of Mathematical Entities: Nothing Works

Hillary Putnam published an article called "Philosophy of Mathematics: Why Nothing Works," in 1994, and a strong case can be made that this remains true today (Putnam, 1994). A philosophy of mathematics must answer a number of questions. One is, why does mathematics work in the sciences, and how is mathematics applicable to the world, which was discussed by physicist E. P. Wigner in his frequently cited article, "The Unreasonable Effectiveness of Mathematics in the Natural Sciences (Wigner, 1960). Wigner saw the usefulness of mathematics in the natural sciences as something almost mysterious, and without rational explanation (Wigner, 1960: p. 14). R.W. Hamming has examined a number of responses to this problem, and has argued that these explanations are an inadequate response to the Wigner problem (Hamming, 1980).

To mention just one response, Max Tegmark (2014), proposes that the universe is mathematical, meaning that the physical world is isomorphic to mathematical structures. His Mathematical Universe Hypothesis (MUH) is: external reality is a mathematical structure, of abstract entities and their relationships. As well, he advances the External Reality Hypothesis (ERH): an external physical reality exists independently of humans. Presumably this external reality refers to our reality, and not some other dimension/universe of the multiverse. While the true mathematical structure which is isomorphic to the world has not been discovered, when it is, it will be revealed as a Platonic abstract entity existing outside of space-time. Thus, the world is not just described by mathematics, but is mathematics; as he says, that if two structures are isomorphic, then they are one and the same (Tegmark, 2007). He claims that evidence for ERH is also evidence for MUH. Wigner's problem of how to explain the "unreasonable" effectiveness of mathematics is solve, Tegmark believes in his system, because if the world itself is a mathematical structure, then the discovery of mathematical patterns is evidence that the world is completely mathematical (Tegmark, 2007).

Tegmark's position has been shown to be circular (Sullivan, 2014). However, the principal philosophical objection to it, in our opinion, is that if we take Tegmark as being literal in his claim that the world is not just *described* by mathematics, which is true (Hossenfelder, 2020), but *is* mathematics, then physics concerned with an external reality is impossible, since there is no external physical reality as such, being that which is in space-time. As he asserts, mathematical entities are abstract Platonic entities and structures, outside of space-time. Hence, there is then no such thing as conventional physics, which deals with causally efficacious entities and facts inside space-time. That is not to say that neo-Pythagoreanism is false as a metaphysical position, as all sorts of things can and have been argued for in philosophy. But Tegmark has not shown this, because an argument from the presence and success of mathematics in physics does out show that *all* is mathematics (Jannes, 2009). In passing, it is interesting to speculate that if Tegmark's Platonistic position was accepted, it is still conceptually possible, as examined below, that on an independent philosophical basis, mathematical Platonism should be rejected in favour of, say, fictionalism, namely, the thesis that mathematical entities and structures do not exist, and that mathematical statements are false. It would seem then that nothing exists, an argument for ontological nihilism.

Another key issue in the philosophy of mathematics relates to the alleged necessity of mathematics, that mathematical statements are true in some sense quite independent of the existence of physical reality, and perhaps true by necessity, whatever that means (Pap, 1958). Thus, statements in Peano arithmetic, for example, would not be refutable by empirical matters, as such statements are not the sort of things which can be refuted by observations of the world. Yet, even here there is controversy, as argued by Peter J. Lewis, arithmetic seems to apply to macroscopic objects only as an approximation according to spontaneous collapse theories of quantum mechanics (SCTQM) (Lewis, 1997, 2003). The enumeration principle is alleged violated by SCTQM, this being that given n objects, if object O_1 is in a given spatial region, as well as all of O_2, O_3, … to On, then there are in total n objects in that region. Lewis argues that from the position of SCTQM, it can be shown by eigenstate analysis that "it is true of each marble that it is in the box, but it is not the case that there are n marbles in the box" (Lewis, 2003: p. 166). Naturally, this has led to a lively debate that now seems to have fizzled out (Clifton & Monton, 1999; Bassi & Ghirardi, 2001). The technicalities of this debate need not be entered into here; the example shows that it is at least conceivable that mathematical statements, true of the macroscopic world, could be false when applied to the quantum domain (Mortensen, 1989), just as has been proposed in the parallel debate on quantum mechanics violating classical logical principles such as distribution, and hence the need for a quantum logic (Gibbins, 2008).

As another challenge to the view that mathematical statements are necessarily true, or even true, the intuitionist L.E.J. Brouwer (Brouwer, 1967: p. 337) wrote a number of papers producing a critique of classical mathematics (Heyting, 1966: pp. 115-120). Thus, he claimed to be able to be able to prove the following:

> (B) There is a real number R, which is not equal to 0, ~(R=0), as R=0 is contradictory, but it is not provable that R>0, or R<0. (Van Dantzig, 1949)

A number of other counter-examples were advanced, primarily based upon Brouwer's rejection of the classical logical principle of the law of excluded middle (Brouwer, 1967). It is interesting to speculate about how much of classical mathematics would survive his intuitionist philosophy. Nevertheless, as will be discussed below, forms of neo-intuitionism (Hanna, 2022), could offer some progress in addressing the many problems discussed here, but this philosophical worldview will provide no comfort to the reductionist materialist.

Another argument raising doubts that mathematics consists of necessary truths, comes from the paradoxes of set theory, such as Russell's paradox. Consider the set of all sets which are not members of themselves. Is this set a member of itself? If it is, then it is not. If it is not, then it is. There are a number of other set theoretical paradoxes such as the Burali-Forti paradox and Cantor's paradox. Typically, the set theoretical paradoxes have been dealt with by modifying our naïve conception of a set through various formal set theories. Ingenious as these theories have been it would appear from a survey of the critical literature that a final resolution of these difficulties has not been accomplished (Priest, 2006).

For example, Grim argued for some time that the set of all truths, or, all true statements, conflicted with Cantor's power set theorem (Grim, 1984). The power set is the set of all subsets of a given set, and if a set S has n elements, then the Power Set PS has 2^n elements (Suppes, 1960: pp. 46-48). If we take the intuitive idea of a "set" to be a "collection of entities of any sort" (Suppes, 1960: p. 1), then we should be able to meaningfully deal with both the set of all apples, and the set of all true statements. Of course, in the light of the set theoretical paradoxes, logicians have restricted the objects of sets containing special set constituents, as in the set of all sets, and have made a distinction between sets and classes. But set theory should not yield paradoxes merely from considering elements such as sentences, which are ontologically distinct from sets. However, for the set of all truths, for each subset of this set, there will be a truth, and thus a corresponding statement, so there will be at least as many truths as there are elements of the power set, contrary to the power set theorem (or in some systems, power set axiom [Suppes, 1960: p. 46]). Thus, a counter-example is presented to a provable theorem.

Reflecting up this result and other paradoxes of totalities, Rescher and Grim state:

> Set theory was born in paradox, was shaped by paradox, and continues to carry the threat of paradox into its current adolescence. Properly understood … the threat of contradiction is not merely formal and is not to be evaded by merely formal techniques. The fact that there can be no set of all non-self-membered sets might be shrugged aside as a minor logical surprise. Beyond Russell's paradoxical set, however, there are serious philosophical difficulties of coherently conceptualising a set of all things, the realm of unrestricted quantification (or even the sense of restricted quantification), the totality of all events, all facts, all propositions, or all that is true. Sets are structurally incapable of handling any of these. (Rescher & Grim, 2011: p. 6)

If mathematics is taken to be reducible to set theory, then the paradoxes create havoc for the foundations of mathematics. But is this so? This leads us to our next major issue in the philosophy of mathematics

3. Set Theory: Should One Believe?

The core idea of logicism is that all mathematics can be defined in strictly logical terms, and more specifically, in set theoretical terms (Tiles, 1989; Maddy, 1990). Thus, regarding the natural numbers, for example, 0 is alleged to be defined as the set with zero elements, which sounds circular. But "zero elements" are taken to be the null set, so 0 = { } (Halmos, 1974: p. 44). From this, each natural number is equal to the set of its predecessors, that the successor of 7 is 8, but 7 is also a subset of 8, which may be "disturbing" (i.e. counter-intuitive) (Halmos, 1974: p. 45). Usually this is ignored. However, B.H. Slater has argued that numbers are not definable in terms of sets, an idea which is "based on a series of grammatical confusions" (Slater, 2006: p. 59). The general argument is not simply summarized, but his main point is that from grammatical considerations, the empty set is not the number zero; rather the number of elements in the empty set is zero, not that set itself. One argument that can be added here is that to make sense of the empty set, assuming the idea is coherent to start with, one needs to be able to characterize the set. It seems that there can be no non-circular characterization, since defining the set will require the number of zero elements at some point. Hence the set theoretical definition of zero, as the foundation of the natural numbers, will be circular.

And, what is the empty set anyway? What is a set, for that matter? Max Black, in his influential article, "The Elusiveness of Sets" (Black, 1971), notes that for most mathematicians, the term "set" is a primitive concept, implicitly defined by the set theoretical axioms of the system, such Zermelo-Fraenkel (ZF) set theory: whatever satisfies the axioms is a set. That is somewhat disappointing metaphysically, as it is possible that a wide range of objects with divergent ontological properties could be sets, ranging from physical things like animals (Halmos, 1974: p. 1), to abstract objects, even metaphysical, transcendental and theological entities (Black, 1971: p. 616). Of course, simply saying that a set is a collection, class, or aggregate will not do, as these accounts will be circular. Black examines a number of attempts to explicate the concept of a set, and finds them all circular, which is not surprising given how fundamental the concept of a set is, and explications must stop somewhere under pain of infinite regress.

Black favors an alternative account of sets as a form of plural talk, a

> stand-in for plural referring expressions …the word 'set,' in its most basic use [is] ..and indefinite surrogate for lists and plural descriptions. (Black, 1971: p. 631)

Black believed that this plural conception made problematic empty and single-membered pluralities. As he wrote:

> Of course, any transition from colloquial set talk to the idealized and sophisticated notion of making sense of a "null set" and of a "unit set" (regarded as distinct from its sole member) will cause trouble. From the standpoint of ordinary usage, such sets can hardly be regarded as anything else than convenient fictions (like the zero exponent in X^0) useful for rounding off and simplifying a mathematical set theory. But they represent a significant extension of ordinary use. (Black, 1971: p. 633)

Alex Oliver and Timothy Smiley (Oliver & Smiley, 2006) have also expressed skepticism about the notions of singletons and the empty set. They say:

> Our own position is one of skeptical caution. The existence of an empty set should only be accepted if there are strong reasons for doing so. We have looked at the arguments in the literature, and found a dispiriting contrast between the technical virtuosity of set theorists when operating inside their fort, and the general poverty of the arguments offered to persuade others to enter it. (Oliver & Smiley, 2006: p. 127)

They show that many of the received attempts to show that there is an empty set are question begging, assuming at some point in the argument that there is an empty set (Oliver & Smiley, 2006: pp. 127-128).

Most of the philosophical perplexities with set theory lie with the use of not just the potential infinite, but the actual infinite, and the existence of infinite sets. This is one of the main objections to set theory by the Australian mathematician N.J. Wildberger (Wildberger, 2015). Wildberger is critical of the very notion of the infinite, accepting a strict finitism. While he does not discuss questions about the mathematical acceptability of the large cardinal hierarchy, such as inaccessible and hyper-inaccessible cardinals (Kanamori,2003), because he is skeptical about the existence of the ordinary infinities, and the continuum problem (which he views as a *reductio* of the idea of infinite sets), he would also object to the "higher infinities."

Wildberger is skeptical of the coherence of most of the axioms of ZF set theory—they are "awash with difficulties" (Wildberger, 2015: p. 6)—particularly the axiom of infinity, that there exists an infinite set (Wildberger, 2015: p. 6). The axiom is more precisely stated to be: there exists a set containing 0 and the successor of each of its elements (Halmos, 1974: p. 44):

(AI) $(\exists A)$ $[0 \in A \; \& \; (\forall B) \; (B \in A \rightarrow B \cup \{B\} \in A]$.

This axiom was controversial when it was introduced (Ramsey, 1926), but is now standardly accepted by most of those who accept that the concept of an infinite set is coherent. Strict finitists like Wildberger reject the axiom and may note with Zermelo himself that elementary number theory can be done without the axiom of infinity (Suppes, 1960: p. 139). As Wildberger says,

> Do you really think you need to have all the natural numbers together in a set to define the function $f(n) = n^2 + 1$? Of course not—the rule itself, together with the specifications of the kinds of objects it inputs and outputs is enough. (Wildberger, 2015: p. 10)

Indeed, Wildberger is skeptical about an axiomatic approach to mathematics in general: "Axiomatic systems strongly misrepresents the practical reality of the subject" (Wildberger, 2015: p. 7). Wildberger writes:

> The difficulty with the current reliance on "Axiom" arises from a grammatical confusion, along with the perceived need to have some (any) way to continue certain ambiguous practices that analysts historically have liked to make. People use the term "Axiom" when they really mean *definition*. Thus, the axioms of group theory are in fact just definitions. We say exactly what we mean by a group, that's all. There are no assumptions anywhere. At no point do we or should we say, "Now we have defined an abstract group, let's assume they exist." Either we can demonstrate they exist by constructing some, or the theory becomes vacuous. Similarly, there is no need for

> "Axioms of Field Theory," or "Axioms of Set theory," or "Axioms" for any branch of mathematics—or for mathematics itself! (Wildberger, 2015: p. 8)

Wildberger's anti-foundationalism in mathematics is similar in conclusion to Wittgenstein's anti-foundationalism and finitism (Wittgenstein, 1967; Rodych, 2000), with both thinkers rejecting the *actual* infinite; there are no infinite sets such as the set of all natural numbers or the set of real numbers (Rodych, 2000: p. 286), and both are critical of transfinite set theory in particular. Thus, Wildberger writes regarding the Continuum Hypothesis (i.e., there is no infinite set whose cardinality is strictly between that of the integers and real numbers),

> If you have an elaborate theory of "hierarchies upon hierarchies of infinite sets," in which you cannot *even in principle* decide whether there is anything between the first and second "infinity" on your list, *then it's time to admit that you are no longer doing mathematics*. (Wildberger, 2012: p. 9; see also Cohen, 1963)

Wildberger on this point is joined by Solomon Feferman, who does not believe that the Continuum Hypothesis is a definite mathematical problem since

> the concept of an arbitrary set essential to its formulation is vague or undetermined and there is no way to sharpen it without violating what it is supposed to be about. (Feferman, 2011-2012: p 1)

Indeed, Robert Hanna has independently reached the same conclusion, but not with a negative take on set theory and the Continuum Hypothesis, rather that mathematics and logic cannot avoid philosophical commitments, and that metaphysical orientations may impact upon what is acceptable as being mathematics. He argues from a neo-Kantian position the following claim regarding the Continuum Hypothesis:

> According to [my view], the real number structure is logico-mathematically a priori constructible from the set of all consciously experienceable points and stretches in spacetime, together with the set of all possible degrees of any consciously experienceable sensory quality, for each consciously experienceable point or stretch in spacetime. What I mean is that it is an a priori fact about the nature of human experience that any set of points or stretches of experienceable spacetime can instantiate any [possible] degree of some or another sense-experienceable quality. Building on that a priori fact, [my] proposal is that for each distinct point or stretch in sense experienceable spacetime, of which there are a denumerably infinite number, we can also find a denumerably infinite number of different [possible] degrees of some or another sense-experienceable quality. Then we can think of the latter cardinal number as an exponent of the former cardinal number in an operation that yields the former's power set—the set of all its subsets. The cardinality of the result of that power set operation is the same as the first transfinite number, aleph1, which in turn has the same cardinality as the real number … Putting the same point in specifically Kantian

> terminology taken from the first Critique, [I'm proposing] that the basic structure of the continuum is the non-empirical extensive quantity structure as described in The Axioms of Intuition (CPR A162–166/B201–207) insofar as it is also exponentiated, according to the power set operation, by the non-empirical intensive quantity structure as described in The Anticipations of Perception (CPR A165–176/B207– 218). In this sense, the basic structure of the continuum is the Kantian synthesis of the extensive quantity structure and the intensive quantity structure. (Hanna, 2015: p. 391, bracketed material added)

This leads us to the next topic of investigation for controversy and "mystery," the metaphysics and ontology of mathematical entities.

4.Metaphysics and Ontology

The standard debate in the metaphysics and ontology of mathematics is between realism and anti-realism. Mathematical realism "is the view that our mathematical theories are true descriptions of some real part of the world" (Balaguer, 2009: p. 36), however "world" is defined. Mathematical anti-realism is the position that mathematical realism is false (Balaguer, 2009). Platonism is the position that mathematical objects, relations, and structures exist, but not in space-time (Gödel, 1947; Resnik, 1997; Shapiro, 1997), and that "our mathematical theories are descriptions of an abstract mathematical realm (Balaguer, 2009: p. 40). Anti-Platonist realism, holds that mathematical theories are true descriptions of spatio-temporal objects, which could be a true description of mental objects (mathematical psychologism), or physical objects (mathematical physicalism) (Balaguer, 2009: p. 37).

The most commonly held anti-realist position is fictionalism/non-factualism (Balaguer, 2021), which holds that mathematical objects, structures and relations do not exist, so any existential mathematical statement is strictly false and mathematical singular terms are vacuous (Balaguer, 2009, 47, 2021; Field, 1980,1989; Hellman, 1989, 1996; Chihara, 1990; Sober, 1993; Yablo, 2001; Azzouni, 2004; Sanchez-Bennasar, 2014, 204; Maddy, 1997, 2005). Thus, as there is no such thing as the number 1, and the equation, 1 + 1 = 2, for example, is false.

Balaguer in *Platonism and Anti-Platonism in Mathematics* (Balaguer, 1998), and in a more recent overview paper (Balaguer, 2009), has given an outline of the principal arguments against both realism and anti-realism in the philosophy of mathematics, and he concludes with his own interesting, but we will argue, logically flawed position. He concludes that there is only one version of realism, full-blooded Platonism (FBP), which survives the standard criticisms, and one version of anti-realism (fictionalism), and that there are no good reasons for choosing one of these over the other. We will argue that this constitutes a case against both of these positions, and as he says, this may be thought to "undermine the two views" (Balaguer, 2009: p. 35).

5. Mathematical Fictionalism

Let us begin with a consideration of fictionalism. The idea that mathematical objects do not exist, so that sentences like "1 + 1 = 2" are false, or at least not true, flies in the face of the way mathematicians view their subject matter. It also is inconsistent with psychological research indicating that there is a biological origin to at least arithmetic, an expression of the "deep structure'" of human perception (Grace et al., 2023, 2024). Thus, one attempt to deal with this objection is Meinongianism, the view that mathematical theories produce true descriptions, but these objects do not exist, but there are still true mathematical statements. In more recent times, this position has been defended by, as he was then known, Richard Routley (Routley, 1980) and Graham Priest (Priest, 1983, 2003, 2005). The position holds that even sentences involving contradictory objects can be true. Thus, "the round square, is round and square," is true, but "the round square is triangular," is false. Balaguer objected to the Meinongian account, that "Fa" can be true, even if there is no object denoted by "a," since in the standard account of truth, if there is no object, then the sentences "Fa" is not true (Balaguer, 2009: p. 49). But this begs the question against Meinongianism.

Routley, in *Exploring Meinong's Jungle*, defends *noneism*, that everything is an object of discourse, but abstract entities do not exist. He rejects the "reference theory," the ontological assumption that all proper subject terms of true statements must have an actual reference, that is that the truth of *Fx* implies that *x* exists (Routley, 1980: p. 24). Routley's *Independence Thesis* is that "items can and do have definite properties even though nonentities" (Routley, 1980: p. 28). Being is not part of the characterization of an object; unlike in existentialism, essence precedes existence. But what is, or was existence for Routley? He does not give a definition of "exists," yet does say that there is a stable meaning of the word. However:

> [A]s the Platonist hastens to point out, there is nothing to prevent us using the word "exists" in any way we please. However, it is not unreasonable to require that before we adopt a usage which is misleading and liable to cause dislocation and equivocation, it should be clear that the gains for doing so at least outweigh the losses. (Routley, 1980: p. 634)

Routley discusses the meaning of existence in chapter 9 of *Exploring Meinong's Jungle and Beyond*. He shows at length that all of the standard accounts are unsatisfactory, often being circular, which is to be expected with such a fundamental concept (Routley, 1980: p. 701) He concludes that "No rock-hard criterion for what exists has appeared" (Routley, 1980: p. 730). This makes it philosophically problematic to then claim that abstract objects, such as sets do not exist, as Platonism supposes (Routley, 1980: p. 730).

Further, Routley did not produce an account of mathematical truth which was coherent with his Meinongianism. Indeed, he rightly notes that there is a circularity problem with the truth definition, “Statement A is true in selected framework M”:

> [I]n order that the definition of “statement A is true in selected framework M” shall provide us with a definition of truth, of “A (of L) is true” is that the framework M selected is the *correct* one, one whose base world T is the factual world, and represents the class of true statements. Thus, in order to define truth, we have already, in effect, to be supplied with the class of truths, T. An irremediable circularity thus appears to have crept into the business of giving a semantical definition of truth. (Routley, 1980: p. 330)

Routley then concludes that each theory of truth can furnish its own criterion for determining M; e.g., for the coherence theory, M is the model that coheres with experience (Routley, 1980: p. 331). And, “every theory is correct according to its own lights” (Routley, p. 334). The corresponding pluralism adopted by Routley, when he changed his name to Richard Sylvan, in *Transcendental Metaphysics* (Sylvan, 1997), means that he has no reason to convince Platonists to abandon their position, and no independent reason for accepting Meinongianism.

Priest gave an alleged anti-realist account of mathematical truth (Priest, 1983), which we will now consider. We will not discuss his own version of Meinongianism, modal noneism (Berto, 2014), since even if this position does give a satisfactory account of the logic of non-existence and objects, it still does not show by its “existence” that, say, mathematical Platonism is false, because there could be independent arguments for that position quite apart from referential considerations, such as the practice of working mathematicians via their usage, and possible indispensability (Baker, 2001).

Priest set out to give the truth conditions for mathematical statements, without the use of abstract entities. He illustrated this approach with a discussion of both arithmetic and set theory, but we will consider arithmetic only (Priest, 1983: pp. 50-52). Priest first defines the language of arithmetic, the signs and terms of the language. There is one constant 0, a one-place function S, and the two-place functions + and x. Terms are formed recursively from 0 using S, +, and x. A numeral is any term with a string of S’s preceding a 0. Atomic formulas are of the form “$t_1 = t_2$,” where t_1 and t_2 are terms. The absurdity symbol is *f*. The set of formulas *F* is the closure of the set of atomic formulas, with “the standard truth conditions for first-order languages with the implication operator and substitutional quantifiers” (Priest, 1983, 51). The truth conditions for atomic sentences are then given using canonical forms:

The canonical form t^* of a term *t* is a numeral which can be given a recursive definition as:

$0^* = 0$

$(St) = St^*$

$(t_1 + t_2)^*$ = the term obtained by prefixing all the S's at the beginning of t_1^* to t_2^*

$(t_1 \times t_2)^*$ = the term obtained by replacing every occurrence of S at the beginning of t_2^* by as many S's as commence t_1^*.

The truth condition of '$t_1 = t_2$' is:

'$t_1 = t_2$' is true iff t_1^* is the same as t_2^*.

That is all fine, but what seems to be missing is an account of the truth of the atomic sentences to get the show going. Priest writes "[f]irst let us suppose that we have an account of the truth conditions of atomic sentences" (Priest, 1983: p. 51). However, let us *not* suppose this, and ask instead: how are atomic sentences of arithmetic shown to be true? One suggestion is to proceed inductively showing that sentences such as "1 + 1 = 2" is true, "1 + 2 = 3" is true, '1 + 2 = 3' is false and so on (Priest, 1983: p. 52). But that will require an infinite number of proofs, which cannot be done. Priest then suggests taking say the Peano axiom system, and showing that all its theorems are true under his proposed truth conditions (Priest, 1983: p. 52). But that too will involve an infinite number of proofs, which will not work. Finally, Priest suggests what he thinks is the better approach:

> The realist has a way of specifying precisely these sentences of arithmetic which are true, viz., those sentences which hold in (that mathematical fiction) the "standard model." Now it is easy to prove by simple induction over formation that a sentence of arithmetic is true under the truth conditions I have given iff it holds in the "standard model." Thus, we can argue *ad hominem* against the realist that the truth conditions are right. If his account fits the pretheoretic data, so does ours. (Priest, 1983: p. 52)

However, for our purposes, which is to conduct a neutral evaluation of *all* these positions, the *ad hominem argument* will not be convincing, since the truth of mathematical realism is as much under critical scrutiny as Priest's alternative. We thus do not ascertain in any non-question-begging fashion, the method of proving the truth of the atomic sentences of arithmetic.

Feng Ye (Ye, 2010, a), has proposed that physicalism about cognitive subjects implies mathematical nominalism. If physicalism is true, there will be a complete physical description of everything in the world consisting of a mathematician doing mathematics. In proving a statement, "there are an infinite number of primes,"

> [w]e won't say anything like "our brains are committed to the existence of numbers." Our acceptance of the sentence is also a physical event involving some neural activities in our brains and a neural circuitry (that associated with the sentence interacting with other physical things. (Ye, 2010a: p. 134)

As we see it, however, even if physicalism is true, it does not imply mathematical nominalism. There is no demonstration that abstract mathematical entities do not exist merely because physicalism about the world, including the mind, is supposedly true. And, if there were the implication Physicalism → Nominalism, then it could be shown that nominalism was unacceptable for some independent reasons, then we could then reject physicalism, which is incorrect, the position could not be refuted in that way. There could be good philosophical reasons for physicalism about the world, but also reasons against nominals, in fact in another paper (Ye, 2010, b), Ye says that mathematical anti-realism must in general explain: (1) what, if anything exists in mathematics; (2) the objectivity of mathematics; (3) the prima facie apriority and necessity, and (4) how mathematics is applicable to reality (Ye, 2010,b, 15). And he concludes: "No current anti-realistic philosophies can meet all these challenges and constraints (Ye, 2010, b, 15). We accept that conclusion and move now to consider mathematical realism.

6. Mathematical Realism

The main problem with mathematical realism is that if mathematical entities do not exist in space-time, then there is no causal interaction between these entities and the human mind, which is the famous Benacerraf objection (Benacerraf, 1973). He presupposed a causal theory of knowledge, but the same point can be made without that, which is that for Platonism, there is a difficulty, indeed a *mystery* as to how reference, and understanding of mathematical entities is possible, as Platonic entities are outside of space-time, but the human mind is not (Hodes, 1984). Here the appeal is usually made to mathematical intuition, which somehow without contact generates knowledge. But that rational faculty is seen by critics as "mysterious." Now, the point of this paper is to recognise just that, that the queen of the sciences, has mystery at its heart, so given that, we should not be disturbed when we find the mind as a mystery as well.

Balaguer has argued that a Full Blooded Platonism (FBP), where "all the mathematical objects that possibly could exist actually do exist" (Balaguer, 2009: p. 59) can solve the epistemological problem noted without any sort of information-transferring contact, between human minds and abstract objects:

> Since FBP says that all the mathematical objects that possibly could exist actually do exist, it follows that if FBP is correct, then all consistent purely mathematical theories truly describe some collection of abstract mathematical objects. Thus, to acquire knowledge of mathematical objects, all we need to do is acquire knowledge that some purely mathematical theory is *consistent*. (It doesn't matter how we come up with the theory; some creative mathematician might simply "dream it up."). But knowledge of the consistency of a mathematical theory—or any kind of theory, for that matter—ddoes not require any sort of contact with, or access to, the objects that the theory is about. Thus, the Benacerrafian lack-of-access problem has been solved; we can acquire knowledge of abstract mathematical objects without the aid of any sort of information-transferring contact with such objects. (Balaguer, 2009: p. 59)

One might first ask how paraconsistent mathematics, based upon the study of inconsistent mathematical objects (Mortensen, 1995) fits into this picture? But leaving that point aside, the problem with the appeal to consistency, that mathematical knowledge requires knowing that some mathematical theory is consistent, runs into the problem of Gödel's Second Theorem: that for any consistent system S, where elementary arithmetic can be carried out, the consistency of S cannot be proven in S. This result applies to all systems upon which ordinary mathematics is based, such as ZFC set theory. So, while there may be consistency proofs of some mathematical theories, the consistency of the *whole* of mathematics is not provable. Even regardless of Gödel's Second theorem, if there was an alleged proof of the consistency of all of mathematics, MM, then one could in turn request a proof that MM is consistent. Appealing to MM itself is circular, and if another method is used MM^1, then an infinite regress is generated (Lakatos, 1962). And finally, if Brouwer is right (Brouwer, 1967), then classical mathematics is inconsistent, which if true, would undermine Balaguer's argument among other things. As well, there is inconsistent mathematics (Mortensen, 1995).

Balaguer has also argued that for FBP and anti-Platonism in the form of fictionalism, there is no good reason for choosing one over the other; there is no fact of the matter about which position is correct (Balaguer, 2009: p. 90). His argument is that first FBP and fictionalism are in agreement on most aspects of their philosophy except the existence of abstract objects. But this disagreement cannot be settled either directly or indirectly. It cannot be settled directly as there is no direct access to the mathematical realm, so there is no way of knowing hat such entities exist. The indirect route of examining the consequences of the positions also fails, as the only material difference is over the question of the existence of abstract entities. And as well, there is no fact of the matter about the existence of mathematical objects existing outside of spacetime. We do not know what a possible world would be like where objects did exist outside of spacetime. If so, there is no fact of the matter as to what possible world counts as there being such objects exist outside of spacetime. And if there is no fact of the matter as to which possible worlds count as to where there exist abstract objects outside of spacetime, then there is no fact of the matter as to the actual world being such a world (Balaguer, 2009: p. 95).

If these arguments are accepted, then we would be left with the situation that the question of the existence of abstract objects outside of spacetime is unsolvable. As he says, that may be thought to "undermine the two views" (Balaguer, 2009: p. 35), which seems a reasonable conclusion. And as well, this would support the position stated at the beginning of his paper, that the philosophy of mathematics leads to metaphysical mystery. We are quite prepared to accept this, holding to the position of epistemic humility, that there are limits to human knowledge, and hence unsolvable mysteries that we just live with a brute fact. But there may be an alternative.

7. Hanna's Neo-Intuitionism as a Way Out of the Impasse?

There is an alternative position to these conventional philosophies of mathematics which we mention in concluding, the neo-intuitionist/neo-Kantianism of Robert Hanna (2022b). This is a part of a philosophical worldview that Hanna has been working on for 20+ years, which attempts to use modernised Kantianism to address many previously unsolved philosophical problems, across philosophy and the sciences, to produce a non-reductionistic human science (Hanna, 2024). We see this as a most welcome development, given the creatively arid condition of contemporary professional Anglo-American philosophy. Thus, with respect to the philosophy of mathematics, Hanna's neo-intuitionism avoids the classical metaphysical and epistemological problems of Platonism, because mathematical objects are not abstract entities existing outside of spacetime, but are Kantian objects of rational human sensibility, and the natural numbers, for example, are "just an immanent structure that is fully embedded in the set of manifestly real, directly and veridically perceivable spatiotemporal material objects in nature" (Hanna, 2022b: p. 14). Further:

> The mathematical natural number structure provided by Peano Arithmetic (and Primitive Recursive Arithmetic and /Cantorian Arithmetic) is abstract only in the non-platonic, Kantian sense that is weakly or counterfactually transcendentally ideal. This is the same as to say that this structure is identical to the structure of the Kantian "formal intuition" of time – as an iterative sequence of homogeneous units that is inherently open to the primitive recursive functions – as we directly and veridically cognize it in Kantian pure or a priori intuition, as represented by formal autonomous essentially non-conceptual content. This content, in turn, must be taken together with all the formal concepts and other logical constructions, including specific logical inference patterns such as mathematical induction, needed for an adequate rational human comprehension of Peano Arithmetic (and Primitive Recursive Arithmetic and Cantorian Arithmetic), that we cognize through conceptual understanding or thinking. (Hanna, 2022b: pp. 13-14).

One interesting part of this neo-intuitionism, which avoids the psychologism of Brouwer, is a rethinking of set theory, *sensible set theory* (Hanna, 2022a). Here sets as represented in ZFC set theory are restricted to objects of human sensibility, or sensible objects, restricted in a broadly Kantian way. As we have seen in our discussion of the Continuum Hypothesis above, Hanna believes that the Continuum Hypothesis can be proven within a neo-Kantian framework, which, if the argument is sound, would be a case of a metaphysical argument contributing a significant mathematical conclusion. That will be an investigation for another day. In particular the question of how sensible set theory escapes the new set theoretical paradoxes of Grim merits a discussion.

8. Conclusion: From Mathematics to Mind

We conclude that standard philosophy of mathematics, which is basically a battle ground between mathematical realism and anti-realism, ends in an impasse, with all positions being seen in some way flawed. Thus, we are left with no satisfactory account of the philosophy of mathematics, leaving aside the neo-Kantian response of Hanna. Hence, received philosophy of mathematics is up to its metaphysical neck in "mystery," and if so, then we should not feel any sort of intellectual shame in accepting the mystery of consciousness. But if we follow Hanna's neo-Kantian turn, we are already in the framework of a non-reductive philosophy of mind, so the idea that mind is somehow metaphysically problematic, can also be rejected. Either way, mind can be accepted to be a non-reducible sui generis fundamental aspect of reality (Hanna, 2024: pp. 205-217). And thus, so are mathematical entities, mystery, or no mystery. Mathematics and mind are essentially intertwined.

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