Mathematical Skepticism: Infinity and Real Numbers

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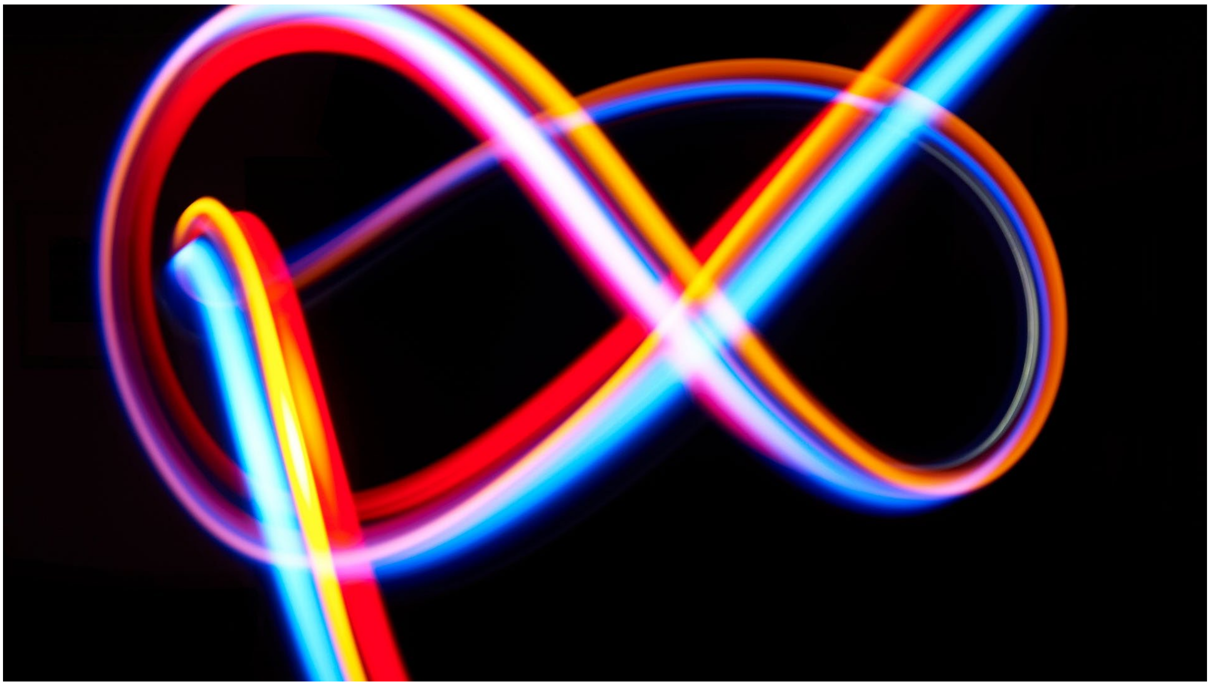

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> In terms of the foundations of mathematics, my position (point of view) is based on the following two main principles (or opinions): (1) No matter which semantics is applied, infinite sets do not exist (both in practice and in theory). More precisely, any description about infinite sets is simply meaningless. (2) However, we still need to conduct mathematical research as we have used to. That is, in our work, we should still treat infinite sets as if they realistically exist. (Robinson, 1964)

> While others are still trying to buttress the shaky edifice of set theory, the cracks that have opened up in it have strengthened my disbelief in the reality, categoricity or objectivity, not only of set theory but also of all other infinite mathematical structures including arithmetic. (Robinson, 1973: p. 514)

1. Introduction

In this paper, we review the positions of some mathematicians and logicians who are skeptical about the existence of real numbers as standardly defined, along the lines of the opening quotes by Abraham Robinson, primarily due to the problem of the actual infinite. We are not concerned here in detail with the foundations of set theory, particularly transfinite set theory, which was subjected to a controversial critique by Wittgenstein (Rodych, 2000), nor with the more rigorous critique than Wittgenstein's by Rescher and Grim (2011) (see also (Feferman, 1989, 1998). Rescher and Grim noted that set theory still faces paradoxes insofar as the power set theorem, or in some systems, power set *axiom*, while provable, can be subjected to counter-examples which cannot be ruled out as items such as the set of all sets can. Thus, the set of all truths: for each subset of this set there will be a truth, so there will be at least as many truths (defined as not sets) as there are elements of the power set, contradicting the power set theorem. Likewise for the set of thoughts and facts, which should form coherent sets as much as the set of all apples. As Rescher and Grim reflect:

> Set theory was born in paradox, was shaped by paradox, and continues to carry the threat of paradox into its current adolescence. Properly understood … the threat of contradiction is not merely formal and is not to be evaded by merely formal techniques. The fact that there can be no set of all non-self-membered sets might be shrugged aside as a minor logical surprise. Beyond Russell's paradoxical set, however, there are serious philosophical difficulties of coherently conceptualising a set of all things, the realm of unrestricted quantification (or even the sense of restricted quantification), the totality of all events, all facts, all propositions, or all that is true. Sets are structurally incapable of handling any of these. (Rescher & Grim, 2011: p. 6)

We have discussed philosophical problems with sets in another paper (Smith & Stocks,2024) (see also, controversially, (Lin et al., 2008; Zhu et al., 2008a, 2008b, 2008c, 2008d, 2008e, 2008f, 2008g, 2008h, 2008i, 2008j, 2008k; Wildberger, 2015).

Finitism views the very presence of infinity in number theory as problematic, primarily on philosophical grounds, rather than being proven to be explicitly contradictory (Van Dantzig, 1956; Isles, 1992; Van Bendegem, 1994, 2000, 200, 2003 Ultra-finitists reject even potential infinities, such as the notion of a set of natural numbers, holding that numbers must be physically realizable (Yessenin-Volpin, 1970, 1981; Zeilberger, 2015)) and that mathematics can be "reduced to manipulations with a (finite!) set of symbols" (Zeilberger, 2015). In this spirit there have been some fertile attempts to reconstruct classical mathematics, including geometry, metric space theory, complex analysis, Hilbert spaces, analysis, and number theory without the use of the concept of infinity, within the framework of mathematical naturalism and nominalism (Parikh, 1971; Shepard, 1973; Mycielski, 1981; Lavine, 1995; Ye, 2011; Maudlin, 2014). As far as we are aware, a complete reworking of classical mathematics has not been fully completed, so technically it may be said that the spectre of infinity still haunts classical mathematics. That being said, let us now examine some theorists who see difficulties with the classical mathematical conception of real numbers, as suggested in the opening quotes of this paper from Robinson, seeing the notion of infinity raising problems for structures such as the real numbers for example.

2. Chaitin and Borel

Gregory Chaitin has argued that here are computational difficulties with real numbers (Chaitin, 2004). This, he claims, refutes the notion that real numbers underlie the physical furniture of the world. Physical reality may be digital and computational (Chaitin, 2005). The first objection Chaitin cites was made by E. Borel in 1927, Borel's "amazing know-it-all number." Thus, if real numbers are infinite sequences of digits, then all of humanity's knowledge can be encoded in a single number (Borel, 1950). The number is written in binary with the n^{th} bit of the binary expansion giving an answer in a natural language to the question, yes=1, no =0. Borel thought that such a number was "unnatural" and not a real number at all because of its artificiality. So, he concluded, there is no reason to believe that such a number existed (Borel, 1952). That may well be so, but it shows only that the know-it-all number does not exist, *not* that there is therefore a problem with real number theory itself. Further, the claim of the number being merely "unnatural," is not a telling objection to the existence of real numbers, if that is Chaitin's aim. What is "natural"?

Chaitin observed that Borel mentioned another alleged paradox, that of "inaccessible numbers." He asserted that real numbers only exist if they can be defined and expressed in a finite number of words, using a natural language such as French or English. This yields a countable infinity of tests for possible expression. But he then argues, there is a supposed denumerable infinity of reals having measure zero. Thus, there will be via diagonalization, reals that cannot be described, contrary to Borel's existence assumption.

Chaitin develops this argument for uncomputable reals. The set of all possible computer programs is countable. So, the set of computable reals is countable, measure zero. By diagonalization, an uncomputable real can be constructed. The key issue here, Chaitin says, is this: if such real numbers are unknowable, why believe in them? Chaitin reinforces this claim with a version of Richard's paradox, by diagonalization over all nameable reals to produce a nameable, but also unnameable real. The set of reals is uncountable, the set of all possible texts in a natural language such as English is countable. Hence the set of all possible mathematical questions being formulated in a natural language is countable too. So, there are by diagonalization real numbers that cannot be defined and are unnameable. However, this is itself a definition or name of the number, hence a version of Richard's paradox.

In response to this, one counter-argument would be to deny that the set of all possible texts in a natural language such as French is countable, or that there is a countable infinity of texts for possible expression in a natural language. The real numbers occur in a natural language, and we are writing about them now. Thus, nothing prevents natural language sentences being given a real number index, say adding "index expression R," to any sentence, in order to produce a 1-1 correspondence between the reals and these indexed natural numbers, hence defeating he Borel-Chaitin argument which requires natural language sentences to be countable, while the reals are uncountable.

Further, from the perspective of classical mathematics which accepts actual infinities, it could be maintained that the Borel-Chaitin argument is question begging as unknowable reals are nothing more than a product of the infinity of the reals, with the finite nature of mathematicians. For classical mathematics, this is all part of the course … of course.

3. B.H. Slater

Western Australian philosopher and logician, B. H. Slater has rejected the idea that numbers are sets, seeing this position "based on a series of grammatical confusions" (Slater, 2006: p. 59). Thus, the empty set is not the number zero, but rather the number of elements in the empty set is zero, not the set itself. To characterize the empty set requires prior recognition that the set is empty, having no members, which means that the number of elements in the set is zero. So, defining zero in terms of the empty set will be circular, and this is the foundation of the natural numbers according to a number of positions in the philosophy of mathematics. Indeed, Slater believes that set theory does not give a correct account of the use of collective terms in general, terms such as "flock" and "groups" (Slater, 1998: pp.144-156).

Slater rejects the idea that there is a determinate, let alone infinite number of natural numbers. He believes that two alleged "infinite sets," even if put in a supposed 1-1 correspondence, may not have the same cardinality as they may have no determinate number at all (Slater, 2002: p.34). There is no number of the natural numbers, the reals, or of the continuum, (Slater, 2002: pp. 35-39), and no irrational numbers:

> [I]f we define them not in terms of impossible Platonic limits but merely convergent sequences of rational numbers, then we are identifying "irrational" numbers with certain functions, since sequences are functions from the natural numbers. But the description "number" is then strictly a misnomer since a function is not a number, even if each of its values is one. (Slater, 2002: p.38)

Slater has also made an attack upon the Weierstrass classic definition of the derivative in the differential calculus, because it presupposes a non-finitist definition of the real numbers involving infinity (Slater, 2002: pp. 171-177). While these remarks are provocative, we believe that the same points have been argued for in more depth by fellow Australian mathematician N.J. Wildberger, whose work will now be considered.

4. N.J. Wildberger

N.J. Wildberger has given a comprehensive critique of classical mathematics in a number of lectures posted on YouTube. As this breaks somewhat academic conventions, which favors the printed word, referencing is difficult in this format, and URLs for YouTube are annoying to type, we find. Thus, it is more convenient to list the titles of the main lectures in the main text, which an interested reader could type into the YouTube search engine. The lectures of relevance include: "A Socratic Look at Logical Weaknesses in Modern Mathematics"; "Mathematical Space and Basic Duality in Geometry"; "Mathematics without Real Numbers"; "The Mostly Absent Theory of Real Numbers"; "The Decline in Rigour in Modern Mathematics"; "Logical Weaknesses in Modern Mathematics"; "Deflating Modern mathematics; The Problem with 'Functions'"; "Reconsidering 'Function' in Modern Mathematics"; "Modern Set Theory—Is It a Religious Belief System?" "The Continuum, Zeno's Paradox, and the Price We Pay for Coordinates."

Wildberger argues that due to the use of the concept of infinity in classical mathematics, "fundamental concepts of calculus, such as continuity, the derivative and integral, rest on the idea of "completing infinite processes" and/or performing an infinite number of tasks" (Wildberger, 2015a, 2015b; see also 2006, 2012, 2021). His position is that conventional real number theory has conceptually insecure foundations, as real numbers do not exist and neither do infinite sequences. The theory of real numbers was heavily influenced by Cantor's theory of infinite sets, and he sees no justification for the postulation of infinite sets.

Real numbers have been viewed in three main ways: infinite decimals, Dedekind cuts, and Cauchy sequences of rational numbers, and Wildberger believes that all of these views have insuperable problems. Starting with the position that real numbers are infinite decimals, Wildberger's main criticism, apart from seeing the concept of infinity as inherently problematic, is that there is a major problem of how to do operations such as multiplication, division, addition, and subtraction for two (or more) non-periodic infinite decimals. Usually, irrationals such as $\sqrt{2}$ and transcendental numbers such as e, can be manipulated as symbols and left as that, but in general it is not possible to decide if given statements involving operations on infinite decimals is correct, by a program, algorithm or function, and "infinite patterns" may not be characterized by a finite rule.

Wildberger does not mention that the formal construction of real numbers as infinite decimals was undertaken by mathematicians such as Karl Weierstrass and Otto Stolz, but not solved. There are more recent attempts (Fardin & Li, 2021; Richman, 1999; Klazar, 2009; Hua, 2012; Gower, n.d.), which seem to have solved the *formal* issue of definition, but not the practical issue raised by Wildberger, of how actually to do the calculations with *non-trivial* examples. Therefore, while it seems that there can be a formal account of real numbers as infinite decimals, it remains problematic from an applied position. However, in section 7 below we will argue that use of infinite decimals and processes can generate contradictions.

Wildberger rejects the Dedekind cut account of the real numbers. This approach defines a cut of the rationals as an ordered pair <A, B> of sets such that:

(1) A and B are not the null set.
(2) A $\cup$ B= the set of rational.
(3) If x ∈ A and y ∈ B, then $x < y$.

And some add:

(4) A has no greatest element (for any a in A, there exists a' in A such that a < a').

A is the lower class and B the upper class, with every element of A preceding every element of B. Real numbers are sections of the rationals (Suppes, 1960: p. 160). A real number is then defined as a Dedekind cut, where the cut represents the boundary between A and B. For rational real numbers, B has a least element. For irrational numbers, B has no least element, as the cut corresponds to a "gap" in the rationals.

Wildberger's primary objection to the Dedekind cut account of real numbers is that Dedekind cuts rely on infinite sets, which he considers ill-defined. A Dedekind cut partitions the rational numbers into two infinite sets, A and B, where A has no greatest element, and every element of A is less than every element of B. Wildberger argues that specifying such infinite sets requires an infinite process, which is not *constructively* feasible. He asserts that mathematics should be restricted to finite, computable objects, and infinite sets like those in Dedekind cuts lack a concrete, verifiable basis.

He further criticizes the practical utility of Dedekind cuts, particularly for transcendental numbers like π or e. While the cut can be readily given for relatively "straightforward" numbers, defining cuts for numbers without simple algebraic properties involves complex, uncomputable specifications. Wildberger asserts that this makes Dedekind cuts "undecidable" in *practice*, since one cannot algorithmically determine whether a given rational belongs to A or B for arbitrary cuts. This, he argues, undermines their claim to define real numbers rigorously.

Wildberger also questions the philosophical underpinning of Dedekind cuts, claiming that they simply assume the continuum they aim to construct. He echoes concerns raised by some mathematical philosophers that the "gap" between A and B presupposes a real number line, introducing circularity. In Wildberger's view, the reliance on axiomatic set theory—for example,, ZFC—to justify infinite sets is a "sleight-of-hand," avoiding the need explicitly to construct mathematical objects before using them.

However, Wildberger's critique is open to counterarguments. Mainstream mathematicians would argue that Dedekind cuts are a logically consistent construction within set theory, requiring no prior assumption of the real number line. The total order of the rationals suffices to define cuts, and the resulting set of cuts satisfies the axioms of a complete ordered field, uniquely characterizing the reals up to isomorphism. Critics can argue that his rejection of infinite sets is a philosophical stance, not a proof of inconsistency. Wildberger's critique of Dedekind cuts reflects his broader finitist philosophy, emphasizing computability and rejecting infinite objects. While his arguments highlight challenges in specifying infinite sets, they do not invalidate Dedekind cuts within the framework of classical mathematics, since classical mathematics would reject his finitism, which they would argue, begs the question against them.

Much the same critique can be made to the *constructivist* critique of Dedekind cuts. L.E.J. Brouwer and Errett Bishop challenged Dedekind cuts for their *non-constructive* nature. Constructivism requires that mathematical objects be explicitly computable or constructible in finitely many steps. A Dedekind cut, as a pair of infinite sets, often cannot be algorithmically specified, especially for transcendental numbers like e. For example, determining whether a rational q belongs to set A or B for the cut representing e requires an infinite amount of information about e's decimal expansion, which is not computable in practice. Intuitionism, as developed by Brouwer, rejects the law of the excluded middle and emphasizes mathematics as a mental construction. Intuitionists have questioned Dedekind cuts for assuming a completed infinity of rational numbers. In intuitionist logic, a set is not a fixed entity but a process of construction, and the infinite partition of rationals into A and B presupposes a "finished" continuum. For instance, Brouwer argued that the real number line is not a pre-existing object but an evolving construct, and cuts imply a static view incompatible with this philosophy (Heyting, 1971). The problem here is that classical mathematicians will simply reject intuitionism and constructivism, seeing them as limiting mathematics, and claiming that there is no no-circular reason for accepting these positions.

The circularity objection is that the cut definition of the reals assumes a total order on the rational numbers and again the existence of infinite sets, which some finitists argue implicitly relies on a conception of the continuum that the cuts are meant to construct (Parsons, 1990). However, this is not a formal circularity in ZFC set theory, as the circularity concern is mitigated by noting that the rational numbers' total order is sufficient to define cuts without assuming the reals. While it does raise questions about whether Dedekind cuts truly explain the reals or merely formalize an intuitive continuum, the classical mathematician will not be disturbed by this. We will return to a critique of the Dedekind account of real numbers after a discussion of real numbers as Cauchy sequences, also rejected by Wildberger.

The Cauchy sequence approach defines real numbers as equivalence classes of Cauchy sequences of rational numbers, a method developed by Georg Cantor.

> Rational Numbers: Let **Q** denote the set of rational numbers (numbers that can be written as a/b, where a and b are integers and b ≠ 0).
>
> Sequences: A sequence of rational numbers is an ordered list of numbers from **Q**, written as {a_n}, where a_n is the n-th term (n = 1, 2, 3, ...).
>
> Cauchy Sequence: A sequence {a_n} of rational numbers is called a Cauchy sequence if the terms get arbitrarily close to each other as n increases. Formally, for every positive rational number ε > 0, there exists a positive integer N such that for all m, n > N, the absolute difference |a_m - a_n| < ε.
>
> Equivalence Relation: Two Cauchy sequences {a_n} and {b_n} of rational numbers are equivalent if the sequence of their differences converges to zero. That is, {a_n} ~ {b_n} if for every positive rational number ε > 0, there exists a positive integer N such that for all n > N, |a_n - b_n| < ε.
>
> Equivalence Classes: An equivalence class of a Cauchy sequence {a_n} is the set of all Cauchy sequences {b_n} that are equivalent to {a_n} under the relation ~.
>
> We denote this equivalence class by [{a_n}].
>
> Real Numbers: The set of real numbers, denoted R, is the set of all equivalence classes of Cauchy sequences of rational numbers under the equivalence relation ~.

For example, the real number 2/3 is the equivalence class of all Cauchy sequences of rational numbers converging to 2/3. Examples of such sequences include:

> {2/3, 2/3, 2/3, ...}
> {0.666, 0.6666, 0.66666, ...}
> {2/3, 4/6, 6/9, 8/12, ...}

and so on, for "infinity."

The equivalence class [{2/3, 2/3, 2/3, ...}] contains *all* sequences {a_n} such that for every $\varepsilon > 0$ in Q, there exists N (a positive integer) such that for *all* n > N, |a_n - 2/3| < ε.

The set of all such equivalence classes, equipped with appropriate definitions of addition, multiplication, and order, forms a complete ordered field, identified as the real numbers. Completeness ensures that every Cauchy sequence of real numbers converges to a real number, resolving the "gaps" in the rational numbers.

Wildberger's objections to Cauchy sequences mirror his critique of Dedekind cuts but focus on specific issues with sequences and their equivalence classes.

Infinite Processes and Non-Constructivity: Wildberger argues that Cauchy sequences rely on infinite processes, which are not constructively feasible in a finitist framework. Verifying that a sequence is Cauchy requires checking infinitely many terms to ensure all pairs beyond some N satisfy the epsilon condition. For example, a sequence approximating the square root of 2 may appear Cauchy, but confirming this property involves an infinite task, which Wildberger deems mathematically suspect. He insists that mathematical objects must be finitely specifiable and computable, and Cauchy sequences fail this test for most real numbers, especially transcendentals like π or e.

Equivalence Classes as Abstract and Ill-Defined: Wildberger criticizes the use of equivalence classes to define real numbers. Grouping all Cauchy sequences converging to the same limit (for example, different sequences approximating the square root of 2) into a single real number involves an infinite collection of infinite objects. He argues that this abstraction is philosophically problematic, as it assumes the existence of uncountably many such classes without explicitly constructing them. In his view, this reliance on set-theoretic machinery (for example, ZFC) obscures the lack of a concrete foundation for real numbers, making the construction "a house of cards."

Practical Impracticality for Transcendental Numbers: Wildberger highlights the difficulty of defining Cauchy sequences for transcendental numbers like π or e. While algebraic numbers like can be approximated by algorithms, transcendental numbers often lack simple recursive definitions. Specifying a Cauchy sequence for π requires an infinite amount of information about its digits, which Wildberger argues is not practically achievable. He contends that this renders the Cauchy sequence construction "undecidable" in practice, undermining its claim to rigor.

Philosophical Objection to Completeness: Wildberger questions the need for a complete number system that Cauchy sequences aim to achieve by ensuring every sequence converges to a real number. He argues that rational numbers, supplemented by algebraic extensions (for example, treating the square root of 2 as a symbol satisfying $x^2 = 2$), suffice for most mathematical purposes. The insistence on completeness, he claims, introduces unnecessary complexity and philosophical baggage, driven by an unproven assumption that infinite limits are meaningful. Wildberger proposes alternatives like working strictly with rational numbers or finite algebraic fields, as in his "rational trigonometry," which avoids real numbers altogether. He believes these approaches are more concrete and computationally well-grounded.

Mainstream Counterarguments: Mainstream mathematicians defend Cauchy sequences as a logically consistent construction within set theory, requiring no infinite computation to define the equivalence classes formally. The completeness property is essential for analysis, enabling results like the Intermediate Value Theorem, they contend. Critics may argue that Wildberger's finitism sacrifices expressive power for philosophical purity, limiting mathematics' ability to model continuous phenomena. The reliance on equivalence classes is seen as a standard abstraction, not a flaw, and computational approximations (for example, decimal expansions) align well with Cauchy sequences in practice. Infinite processes are seen as foundational, so once more the classical mathematician would reject Wildberger's critique based upon finitism, as question begging.

5. A Benacerraf-Inspired Critique of Real Numbers

Our critique of both the Dedekind cut and Cauchy sequence definitions of real numbers adapts and updates Paul Benacerraf's 1965 argument from "What Numbers Could Not Be" (Benacerraf, 1965), which challenges the identification of numbers with specific set-theoretic objects. Benacerraf argued that natural numbers could be defined as different set constructions (for example, von Neumann ordinals or Zermelo ordinals), but equating these sets leads to absurdities, suggesting numbers are not sets but structural entities. We extend this to real numbers, noting that a real number R can be defined as a Dedekind cut, a set D, a partition of rationals into two subsets, or as an equivalence class of Cauchy sequences, a set C. By the principle of identity, if R = D and R = C, then D = C. Equating these sets is ontologically absurd, because they are fundamentally different kinds of objects, implying that R is neither.

This argument highlights the ontological ambiguity in defining real numbers. A Dedekind cut for a real number like the square root of 2 is a pair of sets of rationals, while a Cauchy sequence is an equivalence class of infinite sequences (for example, sequences like 1, 1.4, 1.414, ...). These are structurally distinct: cuts are partitions of a set, while Cauchy sequences are collections of functions from natural numbers to rationals. Equating them as identical sets is problematic, since their elements and constructions differ. For instance, the set D contains rational numbers, while C contains sequences of rational numbers, making D = C incoherent in set-theoretic terms.

This mirrors Benacerraf's point that multiple set-theoretic reductions of a mathematical object create ontological ambiguity, we would say ontological inconsistency. If real numbers can be "reduced" to different set-theoretic objects, their identity seems arbitrary, suggesting that real numbers are not inherently sets but perhaps abstract entities defined by their structural role in a complete ordered field. This aligns with structuralist philosophies, as advocated by Benacerraf and later by Stewart Shapiro, which view numbers as positions in a system rather than specific objects (Shapiro,1997).

However, the mainstream mathematical response, based in set theory (for example, ZFC), is that Dedekind cuts and Cauchy sequences are not claimed to be identical sets but are isomorphic constructions of the real numbers. Both define systems that satisfy the axioms of a complete ordered field, and category theory shows they are equivalent up to isomorphism. The real number R is not literally the set D or C, but an element in a system defined by either construction. Thus, the identity D = C is not asserted; rather, D and C are different representations of the same abstract entity. This supposedly avoids the ontological absurdity by denying that R is identical to either set in a naive sense.

It could also be objected that our argument assumes a strict set-theoretic ontology, where mathematical objects must be specific sets. Structuralists counter that real numbers are defined by their relations (for example, ordering, and arithmetic operations), not by their implementation. On this view, the choice of Dedekind cuts or Cauchy sequences is a matter of convenience, like choosing different coordinate systems in geometry. The "absurdity" of equating D and C dissolves if real numbers are not sets but placeholders in a structure, as mathematical structuralists propose.

We counter-argue against this critique by maintaining that the claim made by mainstream mathematicians that claiming D and C are isomorphic representations, not identical sets, and that R is an element in the system they define (the complete ordered field), does not address the neo-Benacerraf critique. Using "=" suggests an ontological identity that conflicts with the claim of non-identity; if there is no identity then the classical mathematician is not justified from a logical point of view in using "=". Multiple set-theoretic reductions (undermine the idea that numbers are Dedekind cuts or Cauchy sequences, regardless of any view of them being isomorphic representations; arguably both positions apply, and ontological inconsistency follows regardless.

Without going into much detail of the structuralism debate, we note that there are strong criticisms of the position. Structuralism, as defended by Stewart Shapiro for example, posits that mathematical objects like real numbers are positions in a structure (for example, the complete ordered field), not specific objects like D or C. But structuralism's reliance on abstract structures is ontologically problematic (Hellman, 1989). If real numbers are merely roles in a structure, what is the structure itself? Structuralism seems to require the existence of a system (for example, the set of all Dedekind cuts), which presupposes set theory or another foundational framework, risking circularity or infinite regress. This suggests that structuralism does not resolve the ambiguity of D vs. C but shifts the problem to the ontology of structures.

Penelope Maddy, in *Realism in Mathematics*, notes that structuralism struggles with the identity of mathematical objects (Maddy,1990). If two structures are isomorphic (for example, Dedekind cuts and Cauchy sequences), structuralism treats their elements as identical, but this erases potential distinctions. This implies that structuralism glosses over the ontological differences between D and C, which is a flaw in equating them to R. Defining real numbers as structural roles introduces its own philosophical problems, failing to fully resolve the D = C absurdity.

Mathematical structuralism simply relabels the metaphysical problem of real number identity without resolving it. The structuralist insists that mathematics is about structures, not objects, but never adequately addresses what these structures are, or how we access them.

Structuralism trades objects for structures, but it does not eliminate ontological commitment, it transfers it. Instead of committing to the existence of sets or numbers, it now commits to the existence of vast, often infinite structures like the continuum. But for the finitist, this is precisely the problem. Such structures are neither constructible nor epistemically graspable. They are invoked *in toto*, as already-formed infinite systems.

But what does this mean? From a finitist perspective, you cannot assert the existence of a position without asserting the existence of what occupies it. A "point" on the real number line is meaningless if you cannot finitely specify or verify it. The structuralist posits positions without content—forms without substance.

Structuralists attempt to evade Benacerraf's argument by denying that numbers are sets. But they retain the idea that mathematical entities are identifiable by their role in a system. Yet different set-theoretic models (for example, Dedekind cuts vs. Cauchy sequences) describe the same "role" with different underlying elements. This leads structuralists to claim that only the position in the abstract structure matters.

However, if the "role" of √2 is only ever realized by different proxies, none of which is identical to any other, then no object has been specified. This undermines the claim that structuralism explains identity at all. For the finitist, without explicit, finitely realizable identity, there is no mathematical object. There is only semantic fog. Hence, we have an independent argument standing outside of finitism, that we can use to attack the infinitism of classical mathematics, and thus enable Wildberger's critique to go through.

6. Critique of Other Accounts of Real Numbers

The other major accounts of real numbers—continued fractions, Eudoxus reals, metric completion, axiomatic fields, hyperreals, and choice sequences—rely on sets in classical mathematics, with intuitionism offering a partial exception.

Continued Fractions: Real numbers can be defined via continued fractions, where a number is represented as an infinite expression of the form $a_0 + 1/(a_1 + 1/(a_2 + \ldots))$, with a_0 an integer and a_i positive integers for $i \geq 1$. Formally, this is a sequence of integers (a_n), and the real number is the limit of the sequence of rational convergents. This construction requires set theory, as the sequence is an element of the set of all infinite sequences of integers, and equivalence classes may be used to handle different representations. While computationally elegant for algebraic numbers, it shares the set-theoretic reliance of Cauchy sequences, defining real numbers as infinite objects within a set-theoretic framework.

Eudoxus Reals (Constructive Approach): Inspired by Eudoxus of Cnidus and revived in constructive mathematics, real numbers can be defined as "almost homomorphisms" from the rationals to the integers. A real number is a function $f: Q \to Z$ such that for all rationals p, q, $|f(p + q) - f(p) - f(q)| \leq 1$, with additional boundedness conditions. For example, the square root of 2 is approximated by a function mapping rationals to integers based on their proximity to square root of 2. In constructive settings this avoids equivalence classes but still uses sets, as the function f is an element of a set of functions. In classical mathematics, this is embedded in set theory, though constructive versions emphasize computability, aligning partially with our interest in decimals' computational flavor.

Completion of the Rationals (Metric Space Approach): Real numbers can be defined as the metric completion of the rational numbers under the absolute value metric. This generalizes the Cauchy sequence construction, viewing the reals as the "points" added to make the rationals complete (every Cauchy sequence converges). Formally, this involves equivalence classes of Cauchy sequences or a topological construction, both reliant on set theory to define the space of sequences or points. For example, the square root of 2 emerges as the limit point of sequences like 1, 1.4, 1.414, …. This approach is abstract and heavily set-theoretic, as the completion process requires sets of sequences or ideals in a topological space.

Axiomatic Approach (Complete Ordered Field): Real numbers can be defined axiomatically as the unique (up to isomorphism) complete ordered field, satisfying properties like commutativity, order, and the least upper bound property. This approach avoids explicit construction, focusing on the algebraic and order structure. However, in practice, proving the existence of such a field requires a set-theoretic construction (e.g., Dedekind cuts or Cauchy sequences), as the axioms alone do not specify the objects. Even category-theoretic formulations, which emphasize morphisms, rely on sets to model the field. Thus, this account indirectly depends on set theory.

Non-Standard Analysis (Hyperreals): In non-standard analysis, developed by Abraham Robinson, real numbers are embedded in the hyperreals, a field containing infinitesimal and infinite numbers. Reals are identified as the "standard" elements of this larger structure. This requires set theory, often with additional axioms (for example, the axiom of choice), to construct the hyperreals via ultrapowers or non-standard models of the reals. While philosophically distinct, this approach still defines reals within a set-theoretic framework, as hyperreals are sets of sequences or equivalence classes.

Intuitionist Choice Sequences: In L.E.J. Brouwer's intuitionism, real numbers are defined as choice sequences—processes generating rational approximations that converge, guided by free choices rather than predetermined rules. For example, a sequence for the square root of 2 might be constructed by successively refining rationals (e.g., 1, 1.4, 1.414, …) based on computational choices. Intuitionism avoids classical set theory's completed infinities, treating sequences as ongoing processes. However, formalizing choice sequences often involves sets of partial sequences or lawlike rules, and even intuitionist mathematics uses a weak set-theoretic framework to describe collections. Thus, while less set-dependent, this approach still engages with set-like structures.

All of these accounts rely on sets in classical mathematics, where Zermelo-Fraenkel set theory (ZFC) is the standard foundation. Dedekind cuts, Cauchy sequences, continued fractions, Eudoxus reals, and metric completions all define real numbers as sets or elements of sets (for example, sequences, partitions, or functions). The axiomatic approach requires a set-theoretic construction to prove existence, and non-standard analysis uses advanced set-theoretic tools. Intuitionist choice sequences come closest to avoiding sets by emphasizing processes, but formal treatments often involve set-like collections. Non-set-theoretic accounts are rare, as mathematics typically requires a framework to organize infinite objects, and sets are the dominant tool. Alternatives like category theory (emphasizing morphisms) or type theory (used in homotopy type theory) still rely on set-like structures or universes to model real numbers.

Following the critics of set theory cited in this article, including Rescher and Grim, and Wildberger, *we may therefore reject all these approaches to defining the real numbers as well*.

7. Infinite Decimals and a Contradiction

Finally, we conclude with a reconsideration of the account of real numbers in terms of infinite decimals. We previously concluded that this account does survive the finitist criticisms made against it, most notably, that multiplication and addition are not well defined. However, in other papers (Smith, Smith, & Stocks, 2023a, 2023b), we argued that a contradiction can be produced with this account and the idea of supertasks, completing an infinite number of tasks in a finite time. This notion has been employed to deal with Zeno's paradoxes, and it seems to be the status quo position in this area of philosophy of mathematical physics. Robert Hanna has argued that Zeno's paradoxes can be solved without recourse to super tasks, and we concur (Hanna, 2024). But that being said though, as we noted in our previous paper, supertasks have been employed elsewhere in the philosophy of mathematics apart from dealing with Zeno's paradoxes, so we can assume that their use is legitimate, at least for a *reductio*. Summarized, our argument starts with the standard identity:

1.000… = .999… .

Using a supertask, inspired by Hilbert's hotel, we shift decimals and infinite number of times to produce:

1000… = 999…

numbers that violate the Archimedean property and lead to a contradiction, as from this, it is easily seen that 1=0 is provable. Since mainstream mathematics accepts supertasks, rejecting our supertask is ad hoc. The contradiction suggests the real number system is inconsistent. Thus, we force a dilemma: either reject supertasks or accept real number inconsistency. While there are as Hanna has shown, alternative ways of dealing with Zeno's paradoxes without use of supertasks, there have been decades of work devoted to defending the notion. Here we let it stand for the sake of argument, and conclude that what falls is the account of real numbers as infinite decimals, while previously we directed our attack against supertasks.

The objections can be countered by accusing our critics of begging the question.

An Objection from Supertask Invalidity: Critics might argue our decimal-shifting supertask isn't analogous to Zeno's convergent series or Hilbert's Hotel, as it produces ill-defined numbers (1000…, 999…)

Rejoinder: This begs the question by assuming supertasks are only valid in contexts that preserve real number consistency. Since mainstream mathematics accepts supertasks (e.g., summing infinite series), it must justify why our manipulation is invalid without appealing to the system we're questioning. Hilbert's Hotel, which allows infinite shifts, supports our case—critics can't just cherry-pick which infinity is "okay."

The Objection that 999… is hyperreal: One objection is that 999… is a hyperreal, not a standard real, avoiding the contradiction in the real number field.

Rejoinder: This is irrelevant and question-begging. The infinite decimal account claims to define standard reals (for example, 0.999… = 1). If 999… is a hyperreal, the account fails to consistently define all infinite decimals as reals, proving our point. Assuming 999… isn't a real presupposes the system's consistency, dodging the contradiction we've raised. And it does not matter what one calls 999…, as our argument still goes through.

An Objection from Archimedean Violation: Critics note that 1000… and 999… are non-Archimedean, so they're not real numbers, as the real field is Archimedean by definition.

Rejoinder: We embrace this: "Sure, but since the whole system is inconsistent, what do you expect?" If supertasks produce non-Archimedean numbers, the infinite decimal account generates entities it can't handle, exposing its flaws. Assuming the system is Archimedean begs the question, as our contradiction (1 = 0) challenges the field's coherence.

An Objection from Practical Success: Mainstream mathematicians might argue infinite decimals work in practice (for example, approximating π), so theoretical contradictions are irrelevant.

Rejoinder: This begs the question by prioritizing utility over logical consistency. If the system is inconsistent, practical success is a house of cards. Our skeptical stance demands a foundation free of contradictions, not a pragmatic shrug.

Our approach mirrors epistemological skepticism by refusing to accept assumptions (for example, real number consistency, supertask limits) without *independent* justification, just as according to, say, David Hume's problem of justifying induction. This forces critics to engage with the possibility of systemic inconsistency, or simply join Wildberger and other finitists in rejecting the infinitist approach to real numbers, which amounts to embracing the skeptical position accepted by Robinson in the two quotations used as the epigrams of this paper.

As John Lennon might have said: we hope someday you will join us, and the world will be one … or at least finite, once more!

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