Against Mathematical Dogmatism: A Modern Pyrrhonian Critique of Infinity (In the Manner of Sextus Empiricus)

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$$\infty - \infty = ?$$

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1. Prolegomena: The Pretensions of Mathematical Certainty

The mathematical community presents itself as the guardian of *pure reason*, claiming to have constructed edifices of knowledge that stand beyond doubt or contradiction. Yet when subjected to careful examination in the manner of our Pyrrhonian forebears, these supposedly unassailable structures reveal themselves to be built upon arbitrary foundations and sustained by convenient evasions. As Sextus Empiricus demonstrated in his *Against the Mathematicians* (*Adversus Mathematicos*), those who claim certain knowledge in the mathematical sciences often display, upon closer inspection, fundamental contradictions and unexamined assumptions beneath their authoritative pronouncements (Sextus Empiricus, 1935: VII.1-38). The treatment of infinity within modern mathematics provides a particularly illuminating case study of how dogmatic commitments lead to philosophical incoherence disguised as methodological sophistication.

2. The Initial Perplexity: Infinity Minus Infinity

Consider a simple intuitive scenario: an infinite collection of celestial bodies distributed throughout boundless space. Through some cosmic catastrophe—natural or supernatural—every such body is annihilated, removed from existence entirely. Before this event, we had infinity; after the subtraction of all bodies, we observe none remaining. The natural conclusion appears evident: infinity minus infinity equals zero. This fact is not altered in a scenario where, say, half of the bodies are removed; while an infinite number of planets has been subtracted, and an infinity of planets remain, this in itself is one of the weird things about the infinite which those who believe in it should accept, much like Hilbert's hotel. It does not in itself refute the intuition behind the first thought experiment.

To bolster this point further against the dogmatic dismissal of the intuition, we must emphasize that the "half-removal" counterexample relies on a subtle sleight of hand: it presupposes a particular bijection or mapping between the original infinite set and its subset, which is itself an artifact of set-theoretic conventions rather than an inherent property of infinity. In Hilbert's Hotel, the ability to accommodate more guests by shifting occupants assumes an enumerable infinity (aleph-null), where rooms can be reordered in a countable sequence. But this is not a universal truth about all infinities; it is a consequence of adopting Cantor's transfinite arithmetic, which itself introduces hierarchies of infinities (aleph-null, aleph-one, etc.) to avoid paradoxes. The skeptic need not grant this framework as axiomatic. If infinity is truly boundless and not beholden to such countable reorderings, the annihilation of "all" planets—without specifying a partial or selective mapping—should indeed leave zero, as there are no remaining elements to remap or "shift" into existence.

Moreover, the mathematical rebuttal often invokes the non-uniqueness of results (e.g., $\infty - \infty$ could be 0, ∞, or any finite number depending on the approach), but this variability is not evidence of indeterminacy; it is evidence of the inadequacy of finite arithmetic rules when applied to the infinite. The Pyrrhonist would argue that insisting on indeterminacy is merely a way to preserve the sanctity of those finite rules, rather than adapting or expanding them to accommodate the infinite's peculiarities. For instance, in the planet annihilation thought experiment, the operation is total and uniform subtraction, not a limit of partial subtractions approaching infinity at differing rates (as in the limit examples like $x - x$ vs. $x^2 - x$). Conflating these is a category error: the former is a direct set operation on the infinite, while the latter is a process over finite approximations. If mathematicians wish to reject $\infty - \infty = 0$, they must provide a non-question-begging reason why the total annihilation case should be subordinated to partial or limit-based cases, beyond mere appeal to established doctrine.

Finally, this intuition finds unexpected support in certain non-standard mathematical frameworks that the mainstream often marginalizes. In surreal number theory (as developed by John Conway), infinities and infinitesimals are treated as bona fide numbers with consistent arithmetic, where operations like $\omega - \omega$ (for the smallest transfinite ordinal ω) can be defined contextually without immediate contradiction, though it yields 0 in straightforward subtraction under certain orderings. While surreal numbers introduce their own complexities, they demonstrate that rejecting ∞

- ∞ = 0 is not a logical necessity but a choice rooted in preferring one axiomatic system (e.g., the extended reals) over others. The skeptic thus maintains that the "indeterminate" label is less a discovery of truth and more a protective barrier against exploring alternatives that might undermine the Platonic facade of mathematical certainty.

This intuition aligns with our ordinary understanding of subtraction. When we remove all objects from any finite collection, zero remains. The mathematical establishment, however, rejects this reasoning with characteristic authority. They declare the operation $\infty - \infty$ to be "indeterminate," and refuse to assign it any definite value. Their justification invokes the claim that infinity is not a number in the ordinary sense and therefore cannot be subjected to standard arithmetic operations.

Yet this explanation immediately generates suspicion. If infinity is truly not a number, why do these same mathematicians employ it as though it were one in countless other contexts?

3. The Inconsistency Revealed: Selective Application of Principles

The mathematical treatment of infinity reveals a troubling pattern of selective reasoning. In the extended real number system, infinity is explicitly included as an element and subjected to arithmetic operations according to carefully defined rules:

- $x + \infty = \infty$ for any real number x
- $x/\infty = 0$ for any real number x
- $\infty \cdot \infty = \infty$

These operations treat infinity precisely as though it were a number, complete with deterministic results. The mathematician confidently declares that dividing any finite quantity by infinity yields zero, or that adding infinity to any finite number produces infinity. In calculus, limits involving infinity are routinely calculated using algebraic manipulations that implicitly treat ∞ as a mathematical object subject to transformation.

The same practitioners who perform these operations then turn around and declare that $\infty - \infty$ cannot be evaluated because "infinity is not a number." This represents either a profound confusion or a deliberate inconsistency designed to avoid uncomfortable conclusions.

4. The Evasion Strategy: Indeterminate Forms as Conceptual Shields

When confronted with this apparent contradiction, mathematicians retreat to the doctrine of "indeterminate forms." They argue that expressions like $\infty - \infty$ are indeterminate because different mathematical contexts can yield different results. To illustrate this point, they offer examples:

- The limit of $(x^2 - x)$ as x approaches infinity equals infinity
- The limit of $(x - x)$ as x approaches infinity equals zero
- The limit of $(x + 5 - x)$ as x approaches infinity equals 5

From these varied outcomes, they conclude that $\infty - \infty$ cannot have a single, determinate value.

This argument commits a fundamental error in reasoning. The fact that different limiting processes involving expressions that formally resemble "infinity minus infinity" yield different results, does not demonstrate that the operation $\infty - \infty$ itself is indeterminate. Rather, it suggests that the limit of $f(x) - g(x)$ as x approaches infinity depends on the specific functions f and g, not merely on the fact that both approach infinity.

The confusion arises from conflating the formal operation of subtracting two infinite quantities with the distinct mathematical procedure of evaluating limits of difference functions. These are categorically different mathematical activities, yet the indeterminate form doctrine treats them as identical in order to avoid confronting the possibility that $\infty - \infty$ might indeed equal zero.

Now suppose that we extend the real numbers by a single symbol ∞ and keep:

1. Infinity addition: $\infty + a = \infty$ for every real number a.

2. Infinity subtraction axiom: $\infty - \infty = 0$.

3. Ordinary algebraic laws: associativity of addition and the usual rule $(x + y) - z = x + (y - z)$.

Proof of Inconsistency:

Let r be any non-zero real number (for example $r = 1$).

From (1) we have

$$\infty + r = \infty.$$

Subtract ∞ from both sides:

$$(\infty + r) - \infty = \infty - \infty.$$

By rule (3):

$$(\infty + r) - \infty = r + (\infty - \infty).$$

By (2):

$$r + (\infty - \infty) = r + 0 = r.$$

So the left-hand side equals r and the right-hand side equals 0, giving

$$r = 0.$$

But r was chosen non-zero.

The proof of inconsistency demonstrates that axioms (1) through (3) cannot coexist without leading to a contradiction, for example, deriving $r = 0$ for a non-zero real r. While the mainstream mathematical response is to reject axiom (2) — the assertion that $\infty - \infty = 0$ — this choice is far from inevitable. It presupposes that the other axioms, particularly the treatment of infinity as a manipulable entity akin to a number (as in axiom (1)), and the unqualified application of finite algebraic laws to infinite quantities (as in axiom (3)), are sacrosanct. Yet, the skeptic demands an independent justification for privileging these over axiom (2). The brute fact of inconsistency alone does not suffice, as it merely highlights an incompatibility within the system; it does not diagnose which component is the culprit. To beg the question by assuming axiom (2) must yield, would be circularly to defend the very framework under scrutiny, rather than engaging with the philosophical challenge.

Furthermore, a finitist perspective offers a compelling alternative interpretation: so much the worse for assuming that infinity is an actual entity that can be adjoined to the real numbers in the first place! Finitism, in its various forms, rejects the notion of actual infinity as a completed totality, viewing it instead as a potential or unbounded process that never achieves realization as a fixed object. From this standpoint, the contradiction arising from incorporating ∞ as a symbol with arithmetic properties is not a refutation of $\infty - \infty = 0$, but rather evidence of the incoherence in positing infinity as an "entity" at all. Strict finitists, for instance, argue that mathematics should be confined to finite quantities and operations that can, in principle, be carried out with finite resources, avoiding the paradoxes that plague infinite sets. The planet annihilation thought experiment aligns with this view: if infinity is not a genuine number but a descriptor of endlessness, then "subtracting all" from an endless collection intuitively leaves nothing determinate — zero — without invoking the machinery of transfinite arithmetic, which finitists dismiss as unfounded.

This finitist critique gains traction when considering the historical and philosophical precedents. Aristotelian finitism, for example, permits potential infinity—for example, the endless divisibility of a line—but prohibits actual infinity as metaphysically impossible, arguing that an infinite magnitude cannot exist as a completed whole without leading to absurdities. In modern terms, finitists contend that infinite mathematics introduces entities that are neither empirically verifiable nor computationally feasible, rendering them suspect as foundations for a rigorous discipline. The inconsistency proof, rather than defending the extended real system, underscores the perils of reifying infinity: by treating it as an object subject to finite rules (per axioms (1) and (3)), we invite contradictions that could be avoided by eschewing such extensions altogether. Thus, the skeptic can maintain that axiom (2) captures a valid intuition about total subtraction in boundless contexts, while the real issue lies in the dogmatic insistence on infinity's ontological status. Without independent grounds—beyond pragmatic utility—for prioritizing the infinite framework, the objection fails to undermine the skeptical position.

5. The Deeper Problem: Curry's Paradox and Systemic Inconsistency

The difficulties surrounding infinity represent merely one manifestation of more fundamental problems plaguing mathematical foundations. Curry's Paradox provides an even more devastating illustration of these issues. Sextus, in his critique of geometers and arithmeticians, exposed how mathematical practitioners often adopt contradictory principles when convenience demands it (Sextus Empiricus, 1935: III.40-71). The modern paradox of self-reference reveals similar inconsistencies in contemporary logical frameworks.

Consider the sentence: "If this sentence is true, then all ravens are white." Let us call this sentence C. By the principle of conditional proof, if we can establish that C is true, then we can conclude that all ravens are white. But C asserts precisely that if C is true, then all ravens are white. Therefore, the truth of C immediately entails that all ravens are white.

Since C is either true or false, and we have just demonstrated that its truth leads to the conclusion that all ravens are white, we must either accept this absurd conclusion or admit that our logical system is inconsistent. The sentence C can be constructed to yield any arbitrary conclusion whatsoever, thus rendering the entire logical framework trivial—capable of proving every statement and its negation.

Mathematical logic has not solved this paradox. Instead, it has implemented restrictions designed to prevent its formation. Set theories like ZFC prohibit unrestricted comprehension, while formal systems impose hierarchical constraints to block self-referential constructions. These are not solutions but avoidance strategies—methodological patches designed to prevent the system from confronting its own limitations. Dialetheism, which holds that there are true contradictions of the form A & ~A, attempts to address such paradoxes by embracing paraconsistent logics that reject explosion (*ex contradictione quodlibet*) and weaken principles like contraction or modus ponens to block the paradoxical inference. However, dialetheism does not fully resolve Curry's paradox, because it is not a simple contradiction but a conditional leading to arbitrary conclusions. Thus, it comes down to rejecting various logical principles to save formal logic from epistemic shipwreck, as Graham Priest and almost every other logician does (Priest, 1979). Yet, Curry's paradox is persistent, and as Carrara and Martino have shown (Carrara and Martino, 2011), a version of Curry's paradox arises even from dialetheistically "correct" minimalist principles (weakened conditionals) accepted by Priest. This is a persistent problem with the logical paradoxes; the machinery of modern logic permits ever-new refinements and paradoxes to be produced, escaping existing solutions in a perennial logical arms race.

6. The Pragmatic Defense and Its Philosophical Inadequacy

When pressed on these issues, contemporary mathematicians typically retreat to pragmatic justifications. They argue that their methodological restrictions, while philosophically imperfect, enable the development of useful theories that support science, engineering, and commerce. Mathematics works, they claim, and its practical success validates its methods regardless of abstract philosophical concerns.

This pragmatic defense fundamentally misses the point of the skeptical critique. The issue is not whether mathematics is useful—clearly it is—but instead whether it can sustain its claims to represent pure, consistent reason. The mathematical community presents itself as the exemplar of logical rigor, yet its foundations rest upon arbitrary restrictions and convenient evasions designed to avoid confronting fundamental contradictions.

The pragmatic mathematician resembles a builder who discovers cracks in his foundation but chooses to cover them with decorative facade rather than address the underlying structural problems. The building may remain functional for practical purposes, but it cannot claim the architectural perfection its designer originally promised. The "working mathematician's" solution to foundational problems represents a choice rather than a logical necessity. The decision to exclude self-referential constructions and declare certain operations indeterminate reflects philosophical commitments, not inevitable conclusions derived from pure reason.

7. Epistemic Humility: Mathematics as Human Convention

The examination of infinity and related paradoxes reveals that mathematics, despite its pretensions to absolute truth, rests upon conventional foundations sustained by dogmatic commitments. When confronted with contradictions that threaten its consistency, the mathematical community does not resolve these difficulties through reason alone but instead implements arbitrary restrictions designed to preserve its preferred theoretical structure.

This does not diminish mathematics' practical value or its remarkable achievements. However, it does undermine its claim to represent the pinnacle of trans-social human rational inquiry. Mathematics emerges from this analysis not as the discovery of eternal truths, but as the construction of useful fictions—sophisticated tools for modelling aspects of experience, while carefully avoiding questions that might expose their conventional character.

The modern Pyrrhonist suspends judgment regarding mathematical truth claims while acknowledging the utility of mathematical methods. We need not reject mathematical practice to recognize that it cannot deliver the philosophical certainty its practitioners claim. Like other human institutions, mathematics serves important functions while remaining subject to the fundamental limitations that affect all products of human reasoning.

The treatment of infinity thus serves as a microcosm of broader epistemological issues. When systems of thought encounter internal contradictions, they face a choice: either acknowledge the limitations these contradictions reveal, or implement defensive strategies designed to preserve theoretical coherence at the expense of philosophical honesty. The mathematical community has consistently chosen the latter path, trading intellectual integrity for practical functionality.

From a skeptical perspective, this represents neither success nor failure, but simply another illustration of the human tendency to construct belief systems that serve immediate needs while avoiding ultimate questions about their own foundations. Mathematics remains useful precisely because it does not take its own metaphysical commitments too seriously. This is a pragmatic wisdom that its practitioners might do well to embrace more explicitly.

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