Gödel's Theorems, the (In)Consistency of Arithmetic, and the Fundamental Mistake of Analytic Philosophers of Mathematical Logic

Joseph Wayne Smith, Saxon J. Smith, and N. Stocks

Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I[1]).

Von **Kurt Gödel** in Wien.

1.

Die Entwicklung der Mathematik in der Richtung zu größerer Exaktheit hat bekanntlich dazu geführt, daß weite Gebiete von ihr formalisiert wurden, in der Art, daß das Beweisen nach einigen wenigen mechanischen Regeln vollzogen werden kann. Die umfassendsten derzeit aufgestellten formalen Systeme sind das System der Principia Mathematica (PM)[2]) einerseits, das Zermelo-Fraenkelsche (von J. v. Neumann weiter ausgebildete) Axiomensystem der Mengenlehre[3]) andererseits. Diese beiden Systeme sind so weit, daß alle heute in der Mathematik angewendeten Beweismethoden in ihnen formalisiert, d. h. auf einige wenige Axiome und Schlußregeln zurückgeführt sind. Es liegt daher die Vermutung nahe, daß diese Axiome und Schlußregeln dazu ausreichen, alle mathematischen Fragen, die sich in den betreffenden Systemen überhaupt formal ausdrücken lassen, auch zu entscheiden. Im folgenden wird gezeigt, daß dies nicht der Fall ist, sondern daß es in den beiden angeführten Systemen sogar relativ einfache Probleme aus der Theorie der gewöhnlichen ganzen Zahlen gibt[4]), die sich aus den Axiomen nicht

[1]) Vgl. die im Anzeiger der Akad. d. Wiss. in Wien (math.-naturw. Kl.) 1930, Nr. 19 erschienene Zusammenfassung der Resultate dieser Arbeit.

[2]) A. Whitehead und B. Russell, Principia Mathematica, 2. Aufl., Cambridge 1925. Zu den Axiomen des Systems PM rechnen wir insbesondere auch: Das Unendlichkeitsaxiom (in der Form: es gibt genau abzählbar viele Individuen), das Reduzibilitäts- und das Auswahlaxiom (für alle Typen).

[3]) Vgl. A. Fraenkel, Zehn Vorlesungen über die Grundlegung der Mengenlehre, Wissensch. u. Hyp. Bd. XXXI. J. v. Neumann, Die Axiomatisierung der Mengenlehre, Math. Zeitschr. 27, 1928. Journ. f. reine u. angew. Math. 154 (1925), 160 (1929). Wir bemerken, daß man zu den in der angeführten Literatur gegebenen mengentheoretischen Axiomen noch die Axiome und Schlußregeln des Logikkalküls hinzufügen muß, um die Formalisierung zu vollenden. — Die nachfolgenden Überlegungen gelten auch für die in den letzten Jahren von D. Hilbert und seinen Mitarbeitern aufgestellten formalen Systeme (soweit diese bisher vorliegen). Vgl. D. Hilbert, Math. Ann. 88, Abh. aus d. math. Sem. der Univ. Hamburg I (1922), VI (1928). P. Bernays, Math. Ann. 90. J. v. Neumann, Math. Zeitschr. 26 (1927). W. Ackermann, Math. Ann. 93.

[4]) D. h. genauer, es gibt unentscheidbare Sätze, in denen außer den logischen Konstanten: $\overline{}$ (nicht), $\vee$ (oder), (x) (für alle), $=$ (identisch mit) keine anderen Begriffe vorkommen als $+$ (Addition), . (Multiplikation), beide bezogen auf natürliche Zahlen, wobei auch die Präfixe (x) sich nur auf natürliche Zahlen beziehen dürfen.

Kurt Gödel (1906-1978) and the first page of (Gödel, 1931/1967)

1. Introduction

Storrs McCall, in *The Consistency of Arithmetic*, has presented what he alleges constitutes an "absolute" proof of the consistency of Peano arithmetic, PA (McCall, 2014). This result, if correct, would constitute a major achievement in logic, for as stated at the Oxford University Press website, on the page devoted to McCall's book, ''[a] new proof is given of the consistency of arithmetic, contradicting Gödel's well-known undecidability result of 1931" (OUP, 2023). More precisely, McCall argues that his proof, as being "semantic," can escape Gödel's Second Incompleteness Theorem.

We'll argue in this essay that McCall's consistency proof fails, and that even if it did succeed, it would conflict with Gödel's Second Incompleteness Theorem, and constitute a proof-theoretic paradox. Now, a different approach has been taken by T.

J. Stępień and Ł. T. Stępień, who believe that they have devised a syntactical proof of the consistency of PA, within PA itself, unlike McCall's semantic proof (Stępień & Stępień, 2010, 2017). While a direct critique of this paper cannot be conducted here, the results, if correct, would also undermine Gödel's First Incompleteness Theorem, resulting in a proof-theoretic paradox. Our overall conclusion is that Analytic philosophers of mathematical logic are making a fundamental mistake by trying to show that the consistency of arithmetic is logically independent of Gödel's Incompleteness Theorems.

2. The Background to McCall's Proof

McCall distinguishes between syntactical proof and semantic proofs. "Syntactic proofs" are

> concerned only with symbols and the derivation of one string of symbols from another, according to set rules. "Semantic" proofs, on the other hand, differ from syntactical proofs in being based not only on symbols but on a non-symbolic, non-linguistic component, a *domain of objects*. (McCall, 2014: p. 8)

McCall illustrates the idea of a semantic proof with an example from van Fraassen, using a geometrical diagram to show the consistency of three geometric axioms (van Fraassen, 1980: p. 42). The point to be made is that the idea of "proof" in mathematics is generally wider than that employed by logicians, which is a point well taken.

McCall's proof of the consistency of arithmetic makes use of three-dimensional cube-shaped blocks. A formal semantics, based on this block model, is presented. Section 1 of the paper after which the book is named outlines the formal structure of PA, giving primitive symbols, rules of formation for terms, and the wffs. There are the usual classical logic definitions for implication, the connectives "&" and "v," universal and existential quantifiers, and the definition: $1 =_{df} S0$. Axioms and rules for first order logic (Goodstein, 1961: p. 46; Mendelson, 1964: p. 103) along with eight standard arithmetic axioms (e.g. "(Sx = Sy) → (x = y))," and the induction rule, are given.

As has been said, the formal semantics for PA is based on a block semantics, cubes of homogeneous building blocks. The blocks can be arranged in a linear array, a row with no space between the blocks, or a rectangular array, consisting of rows and columns. For three blocks, A, B, and C, the linear arrays ABC and BCA are indistinguishable. Linear arrays of blocks can be concatenated into a single linear array. Alternatively, a rectangle can be transformed into a linear array by repeated concatenation of the rows, resulting in a linear transformation of the rectangle (McCall, 2014, 11).

The domain of the block model is finite, but there is no upper limit to the number of models. McCall emphasizes that the elements of the block model can be manipulated by "physical transformations," rather than just "symbolic transformations" (McCall, 2014: p. 11).

The semantic block model M = < D, v > has a domain D and an assignment function v, where D is a set of blocks and v connects terms of PA with linear or rectangular arrays of blocks such that the variables x, y, z are assigned a linear or rectangular array of blocks, and 0 is assigned the null set { }. Hence, $x + y$ is the linear concatenation of the arrays x and y, and x x y is assigned the rectangular array having sides $v(x)$ and $v(y)$. The successor of x, Sx, is assigned the linear transformation of what is assigned to x, with one more block.

McCall notes that there is a potential problem with the block model in PA, due to its finite nature. If we consider, for example a 2-block model, then there can be assignments to 0, x, Sx and SSx, but not to SSSx, so that v(SSSx), will be undefined. In a two-block model, v_M (x = SSS0), and v_M (S0 + SS0 = SSS0), will not be assigned the value "true," or the value "false," but will be undefined or indeterminate. Hence, not all theorems of PA will be true in all models. To preserve a bivalent semantics, McCall adopts the system of Abraham Robinson, RPA, using relational predicates such as "S_{um} (xyz) (the sum of x and y is z)," instead of functional predicates like, "x + y" (Robinson, 1965).

RPA has as primitive symbols, the usual logical symbols, variables, constants 0,1, and three-placed relations, S_{um} (xyz) ('the sum of x and y is z'), and P_{rod} (xyz) ('the product of x and y is z). The rules of formation are:

F1: for terms, a, b, and c, S_{um} (abc) and P_{rod} (abc) are wff.
F2: If A is a wff and x is a variable then ($\forall x$)A is a wff.
F3: If A and B are wffs, so are ~A and A&B.
F4: There are no other wffs.

The usual classical logical definitions of the logical connections is given as in PA. Cxy says that the successor of x is y. The axioms and rules of inference for RPA are those of first order logic with identity. There are also the following axioms:

R1: S_{um} ($x0x$).
R2: ~ S_{um} ($x10$).
R3: $S_{um}(xyz)$ & S_{um} (wyz) $\rightarrow x = w$.
R4: P_{rod} ($x00$).
R5: P_{rod} ($x1x$).
R6: $(\forall x)(\forall y)(\forall z)(\exists s)(\exists t)(\exists u)$($S_{um}(yzs)$ & S_{um} (xsu) & S_{um} (xyt) & S_{um} (tzu)).

R7: $(\forall x)(\forall y)(\forall z)(\exists r)(\exists s)(\exists t)(\exists u)(S_{um}(yzr)\ \&\ P_{rod}(xrs)\ \&\ P_{rod}(xyt)\ \&\ P_{rod}(xzu)\ \&\ S_{um}(tus)$.
R8: $(\exists y)[Cxy\ \&\ (\forall z)(Cxz \rightarrow z = y)]$.
R9: $(\exists z)[S_{um}(xyz)\ \&\ (\forall w)(S_{um}(xyw) \rightarrow w = z)]$.
R10: $(\exists z)[P_{rod}(xyz)\ \&\ (\forall w)(P_{rod}(xyw) \rightarrow w = z)]$.
The Induction rule: from $\vdash F(0)$, and $\vdash (\forall x)(\forall y)((Fx\ \&\ Cxy) \rightarrow Fy)$, infer $\vdash (\forall x)Fx$.

The semantic models for RPA M = <D, *v*> are a domain D of blocks, an assignment function *v*, which assigns a linear or rectangular array of blocks to the variables, a single block to the constant 1, and the empty set { } to the constant 0 (McCall, 2014, 13). Moreover:

(1) Two linear arrays are equal if they form a rectangle when placed beside each other.
(2) A rectangular array and a linear array are equal if the latter is a linear transformation of the former; and
(3) Two rectangles are equal if their linear transformations are equal (McCall, 2014: p. 14).

The valuation function v_M with respect to a model M, maps every wff of RPA onto the set of truth values {T, F}, and is defined inductively. For example, $v_M(Cxy)$ = T iff the linear transformation of $v(y)$ is equal to the linear transformation of $v(x)$, plus one block. Truth in a model is defined in the usual way; a wff is true in a model M iff $v_M(A)$ = T. A wff A is valid iff it is true in all non-empty models. Validity is restricted to truth in non-empty models, since ~ (1.0 = 1) is intuitively valid, but for empty models is not true (McCall, 2014, 14-15).

The definition of validity as true in all non-empty models, is used in McCall's semantic consistency proof of RPA. The method, as expected, is to prove that all RPA axioms are valid, true in all non-empty models, prove that the rules of inference preserve validity, so that all of the theorems of RPA will be valid. McCall makes the classical logic assumption that it is impossible for two wff A and ~A to both be assigned T: if one takes T, the other takes F (McCall, 2014: p. 16). McCall alleges that if RPA can be shown to be consistent on this basis, then PA is also consistent, since, as he also shows in his paper, all PA theorems are derivable in RPA. Thus, if all RPA theorems can be proven to be valid, then PA will be proven to be consistent.

3. A Critique of McCall's Proof

McCall's proof proceeds by a case-by-case proof of the validity of the RPA axioms, proceeding by *reductio*. Here we will discuss the proof of axiom R2: ~ $S_{um}(x10)$.

However, before doing that we will raise some metaphysical problems with the block model for arithmetic.

For this type of nominalist approach to the foundations of arithmetic, the block model could be replaced, in principle, by a domain consisting of allegedly indistinguishable elements such as tokens of "~" or any other sign. Nevertheless, whatever physical objects are used, there is a problem in assuming that these objects are indistinguishable, and hence can serve as a model for "one." There are, of course many technical metaphysical problems arising which bear on this issue, such as "the problem of the many," which faces objects such as signs like "~" that lack precise borders, as would be seen by examining the sign under a powerful microscope. Some philosophers see the problem of the many as presenting a case for ontological nihilism about such objects, and that would include the sorts of blocks used by McCall (Unger, 1980).

There are also related problems of mereological nihilism, in the light of an interpretation of quantum mechanics, as raised by Grupp, which would also raise problems for McCall's position, and for most of us, as "only partless fundamental particles exist (electrons, quarks, etc.), they do not compose any composite objects, and thus empirical reality does not exist" (Grupp, 2006: pp. 245-246). Grupp also says:

> These experimental findings of quantum physics show that quantum objects are not the sorts of items that can constitute macroscopic objects—or any objects whatsoever: material constitution is an illusion, and thus everyday ordinary empirical-material reality is some sort of a dream. (Grupp, 2006, 246)

McCall's blocks would, if this position is "correct," not exist either, yet, presumably there would be a mathematical foundation to Grupp's quantum atomism, so McCall's block foundations for arithmetic will in principle be inadequate.

These points are flagged, but would require too lengthy a discussion to do justice to the array of issues raised here. Nevertheless, from the limited discussion, we can see that the simple block model for arithmetic, hides surprising metaphysical issues that would need to be addressed further. This is not to say that this would impose any insuperable objection, merely that the issues would need to be addressed.

What can be said though is that the McCall blocks, assumed to be physical blocks, and not ideal Platonistic objects, are not indistinguishable. There are measurable differences between any two blocks, and probably even the human eye could detect some differences without making measurements using precise scientific instruments, to ascertain say, weight. Consequently, in examining any block, a person individuates it from the rest of the universe, by means of a prior assumption that it is a distinct object. To do this, however, presupposes that the viewing subject already has natural number concepts, and the idea of "one," that the block exists as *one* object, not a collective or mass of objects. If we must already have these concepts to get the entire modelling process going, to even put the blocks together, then the attempt to provide a foundation for arithmetic by use of these individuated material objects, is circular.

Let us consider now the core argument for the consistency of RPA given by McCall, that the axioms are valid, and that the rules of inference preserve validity. It is important to remember, here, that in this essay we are critically considering an alleged "absolute" proof of the consistency of arithmetic. The very nature of considering this issue involves accepting at least the conceptual possibility that arithmetic could be inconsistent. That would be surprising, but if this was so, it is not likely that the inconsistency would be in any way simple. For example, the late Edward Nelson was working on a proof of the inconsistency of PA, which involved technicalities such as the Hilbert-Ackermann theorem, Chaitin's Incompleteness Theorem, and Kritchman and Raz's proof of Gödel's Second Incompleteness Theorem, without use of either diagonalization or self-reference (Kritchman and Raz, 2010; n-Baez, 2011). Nelson's proof was discovered by Terry Tao to be flawed due to a technicality. What is interesting is that the Nelson inconsistency proof got as far as it did, and was not written off as the work of a crank. He might have been correct, and perhaps someone will develop another, this time, correct proof, along similar lines. Therefore, the research program of searching for an inconsistency proof of arithmetic cannot be immediately dismissed.

Thus, in considering McCall's argument, we can rightly demand the very highest standard of rigor, one that we might not demand in other more pedestrian areas of mathematics. For example, we should not grant the assumption of the consistency of arithmetic, for that is what is to be proven, although we would not be pedantic about this if the topic was, say, differential geometry, where foundational consistency issues would be passed by in order to enable mathematical analysis to progress.

Consider now McCall's argument for axiom R2: $\sim S_{um}(x10)$, that the sum of x and 1 is not zero. The proof is by *reductio*:

1. $v_M(\sim S_{um}\, x10) = F$ Assumption
2. $v_M(\sim S_{um}\, x10) = T$ 1
3. From line 2, the concatenation of a linear array, $v(x)$, with a single additional block, equals the null array. As this is absurd, line 1 is therefore false. Hence axiom R2 is true.

The problem with this proof, as intuitively plausible and convincing as it seems, is that in the context of a quest for an "absolute" consistency proof, it is question-begging, smuggling in the very notion of consistency which it seeks to prove. This comes with the *reductio* methodology, which assumes that the existence of an inconsistency means that the contested assumption is therefore false. It is assumed that the principle of non-contradiction is universally correct, namely that p and ~p cannot both be true, so that if p&~p is derived from a premise, then we can conclude that the premise is false. But, in the light of the uncovering of challenging paradoxes such as Curry's paradox, where surprising inconsistencies arise from seeming plausible logical principles, and the entire body of logic and philosophy associated with paraconsistent logic, it is an open question what the correct interpretation of the result is (Priest, 1985-1986, 1989). For all we know, this could be yet another one of the paraconsistent logicians "true contradictions" (Priest, 1979, 1984, 2006). Further, we will be arguing below that McCall's result, if correct, generates an inconsistency with Gödel's First and Second Incompleteness theorems, so that the McCall proof of arithmetic consistency leads to inconsistency itself.

To this it may be said that the block model enables us to "see" the proof before our very eyes. Look, there are the blocks, count them, how could there be an inconsistency! Trust your eyes! Now, once upon a time, that might have been a satisfying argument. But, today, paraconsistent logicians debate the "conceivability," if not actuality, of "true contradictions" in reality, and for some, merely claiming that you do not see any inconsistency, will not be enough. After all, perhaps you are deceived in what you see by a conceptually biased visual system, the product of natural selection and gene survival, not necessarily designed for the seeking of fundamental truth about reality. If 1 = 0, then maybe it is an illusion to suppose that there is *a* block in my perceptual field? (Priest, 1999; Beall & Colyvan, 2001).

Nevertheless, let us now assume that McCall's proof is correct. If this is so, what is the significance of the claim made that McCall's proof "[contradicts] Gödel's well-known undecidability result of 1931" (OUP, 2023)?

4. On Gödel's Second Incompleteness Theorem

Gödel's First Incompleteness Theorem, stated informally, is that in a consistent formal system S, at least strong enough to formalize of PA, there are statements of S, Ӝ and ~ Ӝ, such that neither Ӝ, nor ~ Ӝ are provable. Gödel's Second Incompleteness Theorem, that the consistency of S is not provable in S, i.e., the unprovability of consistency, follows from the First Theorem (Rosser, 1936; Kotlarski, 2004). Again, by way of an informal sketch, consider a sentence false in PA e.g. 1=0, symbolized by Ѝ. Hence S is consistent if ~ Provable_S (⌈Ѝ⌉). However, if S is consistent then this implies the Gödel formula $Ӝ_S$. The provability of the consistency of S implies the provability of $Ӝ_S$, contrary to Gödel's First Incompleteness Theorem.

McCall's proof differs from the alleged proof presented by T. J. Stępień and Ł. T. Stępień, because the latter mathematicians sought to prove the consistency of S within S, which as we have seen is not what McCall set out to do. The proof by Stępień and Stępień would, if correct, not only refute Gödel's Second Incompleteness Theorem, but also the First Theorem as well! This is so because it allegedly proves that there is a false consequence of the theorem. If this is so then the result is indeed extraordinary. Yet, Gödel's First Incompleteness Theorem is provable by a number of methods and it is highly unlikely that there is some as yet undiscovered error in all of the proofs, although there have been (generally regarded as failed) attempts over the years to argue for error in the original Gödel paper (Wette, 1974a, 1974b, 1976; Butrick, 1965; Grappone, 2008). Hence, we have the situation, assuming that T. J. Stępień and Ł. T. Stępień are correct, that Gödel's First Incompleteness Theorem is both valid and not valid, i.e., a paradox. But, does McCall's "semantic proof" avoid this paradox?

McCall has claimed that his "semantic" consistency proof of Peano arithmetic contradicts Gödel's Second Incompleteness Theorem. It does not matter what the source of the proof actually is; what is of logical significance is that there is a contradiction between McCall's result, under the assumption of its being correct, and Gödel's Second Incompleteness Theorem. And, it does not matter if McCall's proof is "semantic" or geometrical, for what is relevant is the contradiction. But, as noted above, the contradiction then spreads to Gödel's First Incompleteness Theorem, leading to the same inconsistency: that this theorem is both valid and invalid.

McCall could argue that his result simply refutes Gödel. However, while this is an arguable position, it seems unlikely given the large number of proofs of the First Theorem in the literature. Therefore, if McCall's proof was correct, then due to this inconsistency, the proof must be incorrect (but also "correct"), hence undermining itself. What would now be produced is a proof theoretic paradox, where both McCall's proof and Gödel's Second Incompleteness Theorem are both proven, so the consistency of PA is provable and not provable, indicating that PA is inconsistent, hence undermining the McCall proof.

Nevertheless, perhaps there are some objections that could be made to Gödel's proofs? At this point it is worth a brief examination of the arguments in philosophical logic by paraconsistent logicians who claim exactly that, that the Gödel formula is indeed paradoxical, and that problems can be avoided by accepting the paradoxical result as a true contradiction.

5. On the Alleged Gödel Paradox

Let us consider S in slightly more formal detail. Suppose that S can represent all recursive functions and where the proof relation is recursive. Then there is a formula

Ӂ (the Gödel formula), such that: (1) if S is *consistent*, Ӂ is not provable in S, and (2) if S is *consistent*, ~ Ӂ is not provable in S (i.e. Ӂ is not refutable), and (3) the axioms and rules of S are intuitively correct, then by an intuitively correct argument it can be shown that S is true, and hence S is true, but undecidable (Boolas, 1989). Here, informal proof is understood by Graham Priest to be non-formalized deductive arguments from basic statements. (Priest, 2006).

According to Graham Priest, Gödel's theorem presents an epistemological problem that has never been adequately dealt with (Priest, 1979, 1984, 2006). Informally, the Gödelian sentence is the sentence: "This sentence is not provable." Assuming that it is provable, then if what is provable is true, then it is not provable. Therefore, it is not provable. However, we have just proved this, so it is provable. Therefore, it is provable and not provable (Priest, 2006). Priest concludes that the Gödelian sentence is a paradoxical sentence, a special case of the semantical paradoxes, so if the standards of proof are inconsistent, then there must be some true contradictions (Priest, 2006).

Let T be a formalization of our naïve proof procedures; there is general agreement by mathematicians that the informal language of mathematics can be formalized, at least in principle (Beall, 1999; Priest, 2006). Then, T satisfies the conditions of Gödel's theorem; that is, if T is *consistent*, then there is a sentence G which is not formally provable or refutable in T, but which can be "proven" to be true by a naïve "proof." Hence, G is provable in T, given what T actually is. Therefore, G is provable and not-provable. Hence, naïve proof methods are inconsistent. Dialetheism allegedly follows because our naïve proof procedures are the very method of argument for establishing the truth of sentences. Therefore, Priest concludes, some contradictions are true (Routley, 1979; Priest, 2006).

There has been constant debate about this argument in philosophical logic circles since Priest presented this argument, with some such as Jc Beall (Beall, 1999) accepting the argument, but many rejecting it, all on different grounds (Berto, 2009; Bura, 2011; Chihara, 1984; Choi, 2017; Shapiro, 2002; Slater, 2016; Tanswell, 2016). As expected, most papers attempt to refute the Gödel's paradox argument, although there is no consensus about where it goes wrong. That is common in Analytic philosophy of mathematics, and indeed in the Analytic philosophy of anything (Dietrich, 2011). Here we will assume that Priest is correct and see whether his argument could be used to reject Gödel's First Incompleteness Theorem, and hence avoid the paradoxes of proof theory, albeit at a high mathematical cost (Schwartz, 2005).

Let us re-examine Priest's Gödelian argument. Why should one accept that this argument shows that naïve proof is inconsistent? Surely, it's because of Gödel's Theorem. However, in his paper "The Logic of Paradox" (Priest, 1979), Priest says that

Gödel's "proof only works if the theory under consideration is *consistent*." Chihara (Chihara, 1984) as well, in his critique of Priest noted that the consistency of T, the formal system that formalizes our intuitive procedures for proving mathematical propositions, is presupposed in Priest's proof that the Gödelian sentence G is not a theorem of T. As well, the consistency of T is presupposed in the metatheoretical arguments for the truth of G.

Chihara did not go on to develop a critical argument against Priest based upon this observation. Gödel's theorem + Priest's argument → (i.e., entails) the inconsistency of the naïve notion of proof. Therefore, the inconsistency of the naïve notion of proof → that Gödel's theorem is not applicable. But, Gödel's theorem *is* applicable because it is generally accepted as a provable and true theorem. In other words, Priest's argument, if accepted, shows that Gödel's Theorem *prima facie* undermines itself in a fundamental way. This self-refuting nature of the theorem cannot be dealt with by the usual paraconsistent mantra of "true contradiction," because the paradox is quite different from the standard logico-semantical paradoxes where we seemingly prove p&~p. Gödel's theorem undermines itself, yet it can be proved, so by Priest's argument, it is a provable falsehood, not a "truth." At a minimum, this means that naïve mathematical proof procedures are *unsound*.

Priest argued against the proposal that naïve mathematical proof procedures are unsound (Priest, 2006). His first counter-argument is that the general objection that there is something wrong with naïve proof procedures without specifying exactly what is wrong amounts to an argument for general (logical) skepticism, and does not address his argument specifically. True, but here we have indeed specified what we believe is wrong with the argument.

Priest also doubts that sense can be made of finding out that our proof procedures are incorrect because the rules of procedure define "correctness," so that once mathematical practices are established, the idea of global mistakes comes to lack clear context. Ironically, that argument sounds exactly like the same sort of question-begging argument used against the very notion of paraconsistent logics in the early period, when the principle of non-contradiction, then a sacred cow, was first challenged by early paraconsistent logicians (Routley, 1979).

Further, as Mortensen has noted (Mortensen, 1981, 1989), contemporary logic has been able to generate counter-models to refute more and more once- intuitive "logical truths" and even counter-models to A → A. Mortensen has argued that it is therefore conceivable that our standard mathematics could be false. Priest (Priest, 1992) also agrees that countermodels can be constructed to *any* arbitrary formula. All that would seem to constitute adequate grounds for believing that logicians can discover that global mistakes can be made, and that we could discover that naïve mathematical proof procedures are unsound. It would certainly give grounds for an epistemological skeptic about logico-mathematical knowledge to challenge Priest's claim that global mathematical mistakes cannot occur.

Indeed, the logico-semantical paradoxes constitute prima facie evidence of the unsoundness of naïve mathematical proof procedures. The paraconsistency approach seeks to avoid accepting that truth preservation may fail in the cases of the paradoxes (such as the liar, strengthened liar etc.), by claiming that the paradoxical conclusion is actually true. It is claimed that this offers a unified, non-ad hoc solution to the logical and sematic paradoxes. But, it has been argued that this paraconsistent path fails for the Curry-style paradoxes, which allow a proof of triviality using minimal logic principles, including in some versions, principles like "naïve proof," used by Priest. (Beall & Murzi, 2013; Carrara & Martino, 2011; Carrara, 2018; Curry, 1942; Estrada-González 2016; Field, 2017; Löb, 1955; Restall, 2007; Shapiro, 2013; Slater, 2014; Tennant, 2004; Van Benthem, 1978). A logico-mathematical skeptic would surely see this as an argument against the epistemological integrity of mathematical logic, which, after all, proclaims itself as the most precise kind of human knowledge.

6. Conclusion

It has been argued here that McCall's proof of the consistency of Peano arithmetic fails. Further, even if it did succeed, that proof, along with the alleged results of T. J. Stępień and Ł. T. Stępień, i.e., an alleged proof of the consistency of PA within PA itself, both contradict Gödel's Second Incompleteness Theorem, and therefore have paradoxical consequences. We briefly considered the debate about whether Gödel's First Incompleteness Theorem leads to paradox. Without endorsing the argument given by Priest, it was shown that even if the argument was correct, paradoxical consequences follow that are not solved by the by-now conventional stance of accepting paradoxes as "true contradictions." If the results of McCall and T. J. Stępień and Ł. T. Stępień are correct, then mathematics does indeed face a new foundational problem in proof-theory. By sharp contrast, our overall conclusion is that Analytic philosophers of mathematical logic are making a fundamental mistake by trying to show that the consistency of arithmetic is logically independent of Gödel's Incompleteness Theorems—hence that the consistency of arithmetic can be proved even if both Theorems are false—a mistake that consists in the vestigial 120 year-old Logicist fantasy, against all the evidence to the contrary, that *somehow or another* all mathematical truths are provable inside *some or another* system of mathematical logic, i.e., the Logicist fantasy of completeness.

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