# Against the Philosophers of Mathematics—*Either* Supertasks, *Or* the Consistency of the Real Numbers: Choose!

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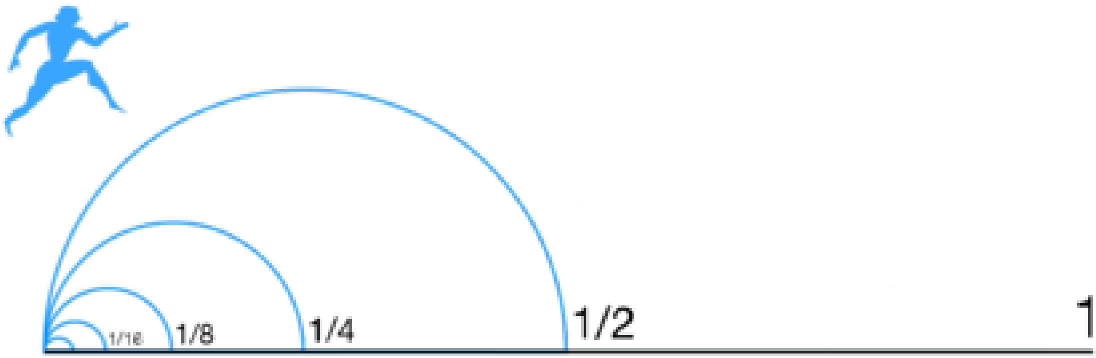


Zeno's "dichotomy" paradox of motion, (supposedly) solved by completing a supertask

## 1. Introduction

Paraconsistent logician Graham Priest has speculated that the natural numbers N could be inconsistent, such that for some natural number n, n = n + 1 (subtracting n from both sides of the equation yields, 1 = 0) (Priest, 1994). He does not prove that this is so, which would be a result of some interest, although some have claimed its truth (Zhu et al., 2008). Priest says that he will make no attempt to give an answer of the form "n is so and so," and "[n]or do I think that an illuminating answer is likely to be forthcoming" (Priest, 1994: p. 3). We take no position here on the question of the inconsistency of the natural numbers, but will argue that given certain philosophical assumptions, it can be proven that the real numbers R are inconsistent, yielding 1 = 0. The core piece of philosophy involves *supertasks*, which most philosophers of mathematics accept as at least conceptually coherent, if not metaphysically possible: to perform a countably infinite number of tasks in a finite time. Indeed, some such philosophical assumption is employed in most contemporary "solutions" to Zeno's paradoxes. We show that philosophers of mathematics who accept the conceptual coherence of the idea of supertasks are faced with the proof of the inconsistency of the real numbers. As the latter option is taken to be unacceptable for most philosophers, except perhaps for some paraconsistent logicians, who thrive on paradox, supertasks must therefore be rejected.

## 2. The Intuitive Proof

The proof of the inconsistency of the real numbers is based upon use of a number consisting of a countably infinite concatenations of 9's:

(P1) 999 …

Suppose for the moment that the number 999 … is mathematically legitimate. This number has some interesting properties. Suppose that we add 1 to it:

(P2) 999 … + 1 = ?

How can this be done, given that the right-hand side is "infinite" and does not have a terminating decimal, although if it did "terminate" in some sense, it would be in a 9 digit, for that is what makes up the number? While this may seem an insuperable difficulty from the standpoint of mathematical Platonism/realism, i.e., the idea that mathematical entities and structures exist in some extra-physical reality as abstract objects/structures, it may not be such a great problem for other philosophies of mathematics, such as nominalism and formalism. From these standpoints, a number such as 999 … is just a string of marks, defined by rules as in a "game" (Goodman & Quine, 1947; Weir, 2010). From the standpoint of mathematical conventionalism (Schroeder, 2018), we have the freedom to define what mathematical concepts we like, how we like, for mathematics is a human symbolic construction governed by pragmatism, not some sort of extra-physical body of eternal truths.

As one of us argued in 1991 (Smith, 1991), the number 999… could be represented by a quasi-terminating notation:

(P3) 999… = 999…999.

We know, all the digits in 999… are 9's, unlike with the expansion of an irrational number √2, or a transcendental number such as π or *e.* The two numbers, 999… and 999…999, can be put into a 1-1 correspondence; for every 9 digit in 999… there is a corresponding 9 digit in 999…999, so having the same informational content, they are equivalent.

Now we can add 1 to 999… to obtain:

(P4) 999…999 + 1 = 1000…000.

But, if we take 1000…000, which by (P4) is greater than 999…, and add another infinite string, 899…999, we obtain:

(P5) 1000…000+ 8999…999 = 999…999.

So, 999… is both greater than, and less than 1000…

That result would likely lead to the mathematicians rejecting the idea that 999… is a real number. Unfortunately, for them, there is a counterargument.

Consider the identity:

(P6) 1.000… = 0.999…

There are various proofs of his (Fraenkel, 1976, 51), with qualifications (Katz & Katz, 2009, 2010 a, 2010 b), with the devil's advocate (Richman, 1999), and dissenters (Smith, 1991). But (P6) is the standard position, and we will run with it here.

For our next metamathematical trick, we make use of the idea of a *supertask*, that is, tasks, consisting of a countable infinite number of steps or operations, that are completed in a finite time (Manchak & Roberts, 2022). In *hypertasks*, the number of operations is uncountably infinite (Clark & Reed, 1984; Al-Dhalimy & Geyer, 2016). The supertask idea goes back to the paradoxes of motion of Zeno of Elea, who supposedly used the idea of the impossibility of performing a supertask to show that motion was impossible, as motion involved traversing an infinite number of steps in a finite time. As contemporary philosopher J. D. Norton has said,

> [t]he resolution of Zeno's paradoxes of motion depended on the idea that completing an infinity of actions is not by itself, an impossibility. (Norton, 2023)

Discussions have continued on this issue, with complexities being introduced via modern mathematics and physics (Benardete, 1964; Earman & Norton, 1996; Salmon, 2001). For example, there is speculation that there can be in principle "supertask computations" to check the truth of conjectures in arithmetic, not yet know (Warren & Waxman, 2020; Manchak & Roberts, 2022). Such computational supertasks would involve hypothetical computers performing a countably infinite number of computations in a finite time (Davies, 2001). While there are, as expected, critics of the idea of supertasks (Gwiazda, 2012), the consensus seems to be that supertasks are theoretically possible.

Relevant to the argument to be given is Hilbert's Hotel, discussed by David Hilbert (1862-1943) in a 1924 lecture (Gamow, 1988). Hilbert considered a hotel with a countably infinite number of rooms, which on day D1 is fully occupied. A traveller comes and wants a room. The receptionist says that even though the hotel is fully occupied, she can get the traveller a room. She moves the first occupant to the second room, the second to the third room, and so on. We are not told how long this shuffle takes, but it does not matter, for even if the shuffles go on for eternity, a spare room is thus found. We believe that the Hilbert shuffle can be applied to equation (P6):

(P6) 1.000… = 0.999…

Let us now perform a supertask involving moving the decimal places of equation (P6) an "infinite" number of times to the right. What type of "infinity" is involved here? The digits in the decimal representation of every real number are countably infinite. This is so, via Cantor, in that the digits of 0.999… and the natural numbers can be put in 1-1 correspondence:

(P7) 0. 9 9 9 9 9 9 9 9 9 9 …
       1 2 3 4 5 6 7 8 9 10 …

There is thus no reason for opposing the supertask of shifting the decimals in (P6) a countable infinite number of times to the right. If this was objectionable then the critic would need to show why such a supposed objection does not undermine other uses of supertasks. Hence performing the supertask of decimal shifting to the right in (P6) leads to:

(P8) 999… = 1000…

But then:

(P9) 999… - 1000… = 0.

And:

(P10) 8999… = 0.

Dividing both sides of equation (P10) by 8999…, yields:

(P10) 1 = 0.

This result is troublesome even for paraconsistent logicians (Priest, 2001, 2006), since 1 = 0, is an absolute inconsistency. Given that, let us now consider objections to this argument.

## 3. Objections and Conclusion

As Freguglia remarks, "once the infinite number is accepted as an infinity, this number does not stay in R" (Freguglia, 2021, 145). Thus, isn't the alleged real number 999… if it is meaningful at all, an infinite hyperreal number $R^*$ (Robinson, 1996; Goldblatt, 1998)? Such infinite hyperreals are the multiplicative inverse of infinitesimals, these in turn being numbers which are non-negative and smaller than any positive real number. Infinite hyperreals are greater than any real number (Robinson, 1996).

However, the existence of hyperreals does not show that 999… is not a real number. An intuitive proof, by means of the notion of a supertask has shown that 999… is in fact a real number. Even if 999… is the "last" real number, just as 000…001, might be considered the "first" real number, there could still be infinitesimals and infinite hyperreals, consistent with the existence of numbers like 999… and 000 …001.

Another objection is that if 999… is to be taken as a real number, then there needs to be a proof that 999… is consistent with the standard axioms of the real numbers (Lay, 2005).

These axioms are as follows:

Field Axioms: there exist conceptions of addition and multiplication, and additive and multiplicative identities and inverses, as:

(A1) (Associative law for addition): $a + (b + c) = (a + b) + c$

(A2) (Existence of additive identity): $(\exists\, 0)$: $a + 0 = 0 + a = a$

(A3) (Existence of additive inverse): $a + (-a) = (-a) + a = 0$

(A4) (Commutative law for addition): $a + b = b + a$

(A5) (Associative law for multiplication): $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

(A6) (Existence of multiplicative identity): $(\exists\, 1)$ $1 \neq 0$ : $a \cdot 1 = 1 \cdot a = a$

(A7) (Existence of multiplicative inverse): $a \cdot a^{-1} = a^{-1} \cdot a = 1$ for $a \neq 0$

(A8) (Commutative law for multiplication): $a \cdot b = b \cdot a$

(A9) (Distributive law): $a \cdot (b + c) = a \cdot b + a \cdot c$

Order Axioms: there exists a subset of positive numbers P such that

(A10) (Trichotomy): exclusively either a ∈ P or −a ∈ P or a = 0.

(A11) (Closure under addition): a, b ∈ P ⇒ a + b ∈ P

(A12) (Closure under multiplication): a, b ∈ P ⇒ a · b ∈ P

Completeness Axiom: a least upper bound of a set A is a number x such that x ≥ y for all y ∈ A, and such that if z is also an upper bound for A, then necessarily z ≥ x.

(A13) (Existence of least upper bounds): Every nonempty set A of real numbers which is bounded above has a least upper bound.

We will call properties (A1)–(A12), and anything that follows from them, elementary arithmetic.

Adding property (P13) uniquely determines the real numbers. (Smith, 2023, slightly modified)

The challenge then is to prove that 999… satisfies each of the axioms, (A1) – (A13). The proof is easy enough for (A2), as 0 + 999…999 =999… 999 + 0. Likewise, proofs can be readily given for (A3), (A4), and (A6). After that, there are difficulties, as multiplying and adding 999… to other arbitrary reals is a grave problem given the infinite decimal expansion. Nevertheless, this is a problem that mathematician Norman Wildberger has discussed in his YouTube videos , "Insights into Mathematics," criticizing the idea of the infinite in mathematics (Wildberger, 2015), and the account of real numbers as infinite decimals. How does one as a matter of practice do operations such as multiplication, division, addition and subtraction for two (or more) non-periodic infinite decimals? Often irrational numbers such as √2, or a transcendental number such as π or *e*, can be manipulated as symbols. But, in the case of infinite incomputable decimals, not given by a program, algorithm, or function, and not characterized by some finite rule, "we cannot in general decide if any given arithmetical statement about such decimals is correct (Wildberger, 2015). As this is a general problem for the conception of real numbers as infinite decimals, not being able to prove that various axioms of (A1) – (A13) hold for 999…, is thus not an objection to 999… being a real number.

But suppose there were a proof that 999… did violate one of more of the axioms (A1) to (A13). This as well would not show that 999… is not a real number. The violation of the axioms would not be surprising since 999… is essentially an inconsistent number, as postulated by Priest (Priest, 1994). And, in any case we have given an intuitive proof via the supertask decimal moving strategy to show that 999… is real. So, at best the critic would only show that 999… is both real and not-real.

Finally, it could be objected that the argument for the inconsistency of the real numbers depends upon the use of supertasks, and it could be argued that since this leads to inconsistency, the method of supertasks should be rejected as illegitimate. We must say that we would be pleased, as ultrafinitists (Yessenin-Volpin, 1970; Feferman, 1989; Ye, 2011), to accept that objection. In fact, it was the aim of the paper to reach that point. Thus, here is a choice: *either* keep supertasks, but accept the inconsistency of the real numbers, *or* keep the consistency of the real numbers, but reject supertasks. Faced with that choice, we are overjoyed to see the notion of supertasks assigned to the philosopher's dust bin.

It's a pity though, because this will mean that Zeno's paradoxes arise once more.

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